DIFFRACTION OF VECTOR ELASTIC WAVES BY A FINE Finite CRACK

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The problem of the diffraction of obliquely incident scalar waves by finite slits and strips with various boundary conditions has been considered previously. Reference may be made to the work of other authors on the diffraction of scalar waves by finite slits. The problem of the diffraction of obliquely incident vector elastic waves by a clamped finite strip has also been treated.

In the present paper we consider the problem of the diffraction of an incident plane harmonic compressional wave by a finite crack. Let the crack be located at \(-1 < x < 1, y = 0, -\infty < z < \infty\) in an otherwise infinite homogeneous elastic medium. The normal components of the stresses at the crack vanish. Reference may be made at this point to a considerable published literature on the diffraction of elastic waves by semi-infinite cracks (or half planes) and allied problems for this geometry.

Let the potential for the incident wave be

\[ \phi_0(x, y)e^{-i\omega t} = e^{i(\alpha x + \beta y - \omega t)} \]  

For this geometry, the problem is seen to be two-dimensional in nature. Let \(u_x e^{-i\omega t}, u_y e^{-i\omega t}\) be the displacement components in the \(x, y\) directions, respectively. Let the scattered fields be derived from two scalar functions \(\phi, \psi\) such that

\[ u = -\nabla \phi + \nabla \times (\psi z_1), \]  

where \(z_1\) is a unit vector in the \(z\)-direction. The equations of elastic wave motion reduce to

\[ \nabla^2 \phi + k_1^2 \phi = 0 \]
\[ \nabla^2 \psi + k_2^2 \psi = 0 \]

where

\[ k_1^2 = \omega^2 \rho / (\lambda + 2\mu), \quad k_2^2 = \omega^2 \rho / \mu, \]

\(\rho, \lambda, \mu\) being, respectively, the density and the Lamé constants of the medium. The components of stress, omitting the factor \(e^{-i\omega t}\), are

\[ \tau_{yy} = \rho \omega^2 \phi - 2pc_2^2 \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y^2} \right) \]
\[ \tau_{xx} = \rho \omega^2 \phi + 2pc_2^2 \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y^2} \right) \]
\[ \tau_{xy} = pc_2^2 \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \right) \]
where
\[ c_{1,2}^2 = \omega^2/k_{1,2}^2. \]

We wish to solve equations (3) subject to the boundary conditions
\[ \tau_{yy} = -\rho \omega^2 \phi_0 - 2 \rho c_s^2 \frac{\partial^2 \phi_0}{\partial x^2}, \quad y = 0, \quad |x| < 1. \]
\[ \tau_{xy} = 2 \rho c_s^2 \frac{\partial^2 \phi_0}{\partial x \partial y}, \quad y = 0, \quad |x| < 1. \] (5)

To these conditions we add the condition that the stresses be of the order \( r^{-c} \) \( 0 < c < 1 \) as \( r \to 0 \), \( r \) being the distance from either edge of the crack. Let
\[ \phi(x, y) = \int_{-\infty}^{\infty} \{ P_1(\zeta) \mp P_2(\zeta) \} e^{i \zeta x - i(r^2 - k_2^2)^{1/2} y} d\zeta \]
\[ \psi(x, y) = \int_{-\infty}^{\infty} \{ \mp Q_1(\zeta) + Q_2(\zeta) \} e^{i \zeta x - i(r^2 - k_2^2)^{1/2} y} d\zeta \] (6)

where the \( \mp \) signs refer to the half planes \( y > 0 \) and \( y < 0 \), respectively. The path of integration for the first integral of (6) is as shown in Figure 1. It is shown below that the functions \( P_1 \) and \( P_2 \) do not have branch points at \( \zeta = \pm k_2 \) in the low-frequency approximation. The analytic function \((r^2 - k_1^2)^{1/2}\) is given a nonnegative real part on the path of integration. The path of integration for the second integral of (6) is defined similarly. We wish to determine the functions \( P_{1,2}, Q_{1,2} \).

From (6) and in view of the continuity of \( \tau_{yy} \), \( \tau_{xy} \) for all \( y \), conditions (5) can be written as
\[ \lim_{y \to 0} \int_{-\infty}^{\infty} \left( \left( \frac{k_2^2}{2} - \frac{k_2^2}{2} \right)(P_1 \mp P_2)e^{-(r^2 - k_2^2)^{1/2} y} + i \zeta_2^2 (r^2 - k_2^2)^{1/2}(Q_1 \
\quad \mp Q_2)e^{-(r^2 - k_2^2)^{1/2} y} \right) e^{i \zeta x} d\zeta = \left( \frac{\partial^2 \phi_0}{\partial x^2} \right)_{y=0} |x| < 1 \]
\[ \lim_{y \to 0} \int_{-\infty}^{\infty} \left( i \zeta_2^2 (r^2 - k_2^2)^{1/2}(\mp P_1 - P_2)e^{-(r^2 - k_2^2)^{1/2} y} + \left( \frac{k_2^2}{2} \right) \right) (\mp Q_1 \
\quad + Q_2)e^{-(r^2 - k_2^2)^{1/2} y} \right) e^{i \zeta x} d\zeta = \left( \frac{\partial^2 \phi_0}{\partial x \partial y} \right)_{y=0} |x| < 1. \] (7)

Since the stresses \( \tau_{yy} \) and \( \tau_{xy} \) are continuous in \( y \) for \( y = 0 \),
\[ \int_{-\infty}^{\infty} \left\{ \left( t^2 - \frac{k_z^2}{2} \right) P_2 + i\eta \left( t^2 - \frac{k_z^2}{2} \right)^{1/2} Q_2 \right\} e^{it\xi} d\xi = 0 \]

\[ \int_{-\infty}^{\infty} \left\{ i\eta \left( t^2 - k_z^2 \right)^{1/2} P_1 - \left( t^2 - \frac{k_z^2}{2} \right) Q_1 \right\} e^{it\xi} d\xi = 0. \]  

Equations (8) are satisfied identically if we set

\[ P_2(\xi) = i\eta \left( t^2 - k_z^2 \right)^{1/2} R_2(\xi) \]
\[ Q_2(\xi) = - \left( t^2 - \frac{k_z^2}{2} \right) R_2(\xi) \]
\[ P_1(\xi) = \left( t^2 - \frac{k_z^2}{2} \right) R_1(\xi) \]
\[ Q_1(\xi) = i\eta \left( t^2 - k_z^2 \right)^{1/2} R_1(\xi), \]

where \( R_1,2(\xi) \) are arbitrary functions satisfying certain integrability conditions. Thus equations (7) take the form

\[ \lim_{y \to 0} \int_{-\infty}^{\infty} \left\{ \left( t^2 - \frac{k_z^2}{2} \right) e^{-\left(t^2 - k_\eta \right)^{1/2} |y|} - \xi^2 \left( t^2 - k_\eta \right)^{1/2} e^{-\left(t^2 - k_\eta \right)^{1/2} |y|} \right\} \]
\[ \times e^{it\xi} R_1(\xi) = \left( \frac{k_z^2 \phi_0}{2} + \frac{\partial^2 \phi_0}{\partial x^2} \right)_{y=0} \quad |x| < 1 \]

\[ \lim_{y \to 0} \int_{-\infty}^{\infty} \left\{ \left( t^2 - \frac{k_z^2}{2} \right) e^{-\left(t^2 - k_\eta \right)^{1/2} |y|} - \xi^2 \left( t^2 - k_\eta \right)^{1/2} e^{-\left(t^2 - k_\eta \right)^{1/2} |y|} \right\} \]
\[ \times e^{it\xi} R_3(\xi) = \left( - \frac{\partial^2 \phi_0}{\partial x \partial y} \right)_{y=0} \quad |x| < 1. \]  

Equations (10) are the integral equations in \( R_1 \) and \( R_2 \) to be solved, subject to the continuity conditions to be imposed along \( y = 0, |x| > 1 \).

The continuity conditions upon \( \phi \) and \( \psi \) at \( y = 0, |x| > 1 \) require that

\[ \int_{-\infty}^{\infty} P_1(\xi) e^{it\xi} d\xi = 0 \quad |x| > 1 \]
\[ \int_{-\infty}^{\infty} Q_1(\xi) e^{it\xi} d\xi = 0 \quad |x| > 1 \]
\[ \int_{-\infty}^{\infty} \left( t^2 - k_z^2 \right)^{1/2} P_1(\xi) e^{it\xi} d\xi = 0 \quad |x| > 1 \]
\[ \int_{-\infty}^{\infty} \left( t^2 - k_z^2 \right) Q_2(\xi) e^{it\xi} d\xi = 0 \quad |x| > 1. \]  

Let

\[ g_n(x) = \int_{-\infty}^{\infty} \left( t^2 - k_z^2 \right)^{1/2} R_n(\xi) e^{it\xi} d\xi, \quad n = 1, 2. \]  

Conditions (11) can be satisfied if \( g_n(x) = 0, |x| > 1 \). Expressions (12) can be inverted to yield
\[ (r^2 - k_2^2)^{1/2} R_n(r) = \frac{1}{2\pi} \int_{-1}^{1} g_n(t) e^{-it\xi} d\xi. \]  

(13)

We use the integral representation for the Hankel function
\[ H_0^{(1)}\left\{ k[(x - t)^2 + y^2]^{1/2} \right\} = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{e^{i(t-x) - (r^2 - k_2)^{1/2}|y|}}{(r^2 - k_2)^{1/2}}. \]

(14)

Then substituting (13) and (14) into (10), we get, after a change of the order of integration,
\[ \lim_{\nu \to 0} \int_{-1}^{1} g_1(t) dt \left\{ \left( \frac{\partial^4}{\partial x^4} + k_2^2 \frac{\partial^2}{\partial x^2} \right) \left( H_0^{(1)}(k_1[(x - t)^2 + y^2]^{1/2}) - H_0^{(1)}(k_2[(x - t)^2 + y^2]^{1/2}) \right) \right\} \]
\[ + k_2^4 \int_{-1}^{1} \frac{\partial}{\partial x} H_0^{(1)}(k_2[(x - t)^2 + y^2]^{1/2}) \right\} = \left( \frac{k_2^2}{2} + \frac{\partial^2 \phi_0}{\partial x^2} \right)_{y=0} |x| < 1 \]
\[ \lim_{\nu \to 0} \int_{-1}^{1} g_2(t) dt \left\{ \left( \frac{\partial^4}{\partial x^4} + k_2^2 \frac{\partial^2}{\partial x^2} + k_2^4 \right) \left( H_0^{(1)}(k_2[(x - t)^2 + y^2]^{1/2}) - \frac{\partial^4}{\partial x^4} \right) \right\} \]
\[ + k_2^4 \int_{-1}^{1} \frac{\partial}{\partial x} H_0^{(1)}(k_1[(x - t)^2 + y^2]^{1/2}) \right\} = \left( \frac{-\partial^2 \phi_0}{\partial x \partial y} \right)_{y=0} |x| < 1. \]

(15)

In order to solve (15), it is sufficient to solve the following equations whose solutions are also solutions of (15):
\[ \frac{i}{2} \int_{-1}^{1} g_1(t) \left\{ \left( \frac{\partial^2}{\partial x^2} + k_2^2 \frac{\partial}{\partial x} \right) \left( H_0^{(1)}(k_1|x - t|) - H_0^{(1)}(k_2|x - t|) \right) \right\} \]
\[ + \frac{k_2^4}{4} \int_{-1}^{1} H_0^{(1)}(k_2|s - t|) ds \right\} dt = \left( \frac{k_2^2}{2} \int_{x_0}^{\infty} \phi_0 dx + \frac{\partial \phi_0}{\partial x} \right)_{y=0} |x| < 1 \]
\[ \frac{i}{2} \int_{-1}^{1} g_2(t) \left\{ \left( \frac{\partial^2}{\partial x^2} + k_2^2 \frac{\partial}{\partial x} \right) H_0^{(1)}(k_2|x - t|) \right\} - \left( \frac{\partial^2}{\partial x^2} + k_2^4 \frac{\partial}{\partial x} \right) \right\} \]
\[ H_0^{(1)}(k_1|x - t|) + \frac{k_2^4}{4} \int_{-1}^{1} H_0^{(1)}(k_2|s - t|) ds \right\} dt = - \left( \frac{\partial \phi_0}{\partial y} \right)_{y=0} |x| < 1, \]

(16)

where for convenience we choose \( x_0 \) such that
\[ \int_{x_0}^{\infty} \phi_0 dx = (i\alpha)^{-1}. \]

We shall use the following series representation\(^9\) for the Hankel function \( H_0^{(1)}(z) \)
\[ \frac{\pi i}{2} H_0^{(1)}(z) = -J_0(z) \log z + \Gamma J_0(z) + 2 \frac{(-1)^m h_m(z/2)^{2m}}{(m!)^2}, \]

(17)

where
\[ \Gamma = \frac{\pi i}{2} + \log 2 - \gamma; \]
\[ \gamma \text{ is Euler's constant}; \]
\[ h_m = 1^{-1} + 2^{-1} + \ldots + m^{-1}, m = 1, 2, \ldots; \text{ and} \]
\[ h_0 = 0. \]
From (17), equations (16) become, after some algebra,

\[ \int_{-1}^{1} \frac{g_1(t)}{x - t} \, dt = \pi f_1(x) - \pi \int_{-1}^{1} F_1(x, t) g_1(t) \, dt \quad |x| < 1 \]
\[ \int_{-1}^{1} \frac{g_2(t)}{x - t} \, dt = \pi f_2(x) - \pi \int_{-1}^{1} F_2(x, t) g_2(t) \, dt \quad |x| < 1, \]  

(18)

where

\[ f_1(x) = \frac{k_2^2 - 2\alpha^2 e^{i\alpha x}}{k_1^2 - k_2^2 \, i\alpha} \]
\[ f_2(x) = \frac{2\beta e^{i\alpha x}}{k_1^2 - k_2^2} \]

\[ F_1(x) = a_{\alpha}^{(0)}(t) + \sum_{n=1}^{\infty} a_{\alpha}^{(n)}(x - t) t^{2n-1} + \log |x - t| \sum_{n=1}^{\infty} \frac{b_{\alpha}^{(n)}(x - t) t^{2n-1}}{i-1}. \]  

(19)

The four sums in \( F_{1,2} \) converge for all \( x, t \). Further, \( F_1 \) and \( F_2 \) are of order \( \omega^2 \) as \( \omega \to 0 \).

We look for solutions \( g_1(x), g_2(x) \) to (18) which are bounded in the interval \([-1,1]\). Here we consider the right-hand sides of (18) as known and apply a formula given by Tricomi\(^{10}\)

\[ g_n(x) = \frac{(1 - x^2)^{1/4}}{\pi} \int_{-1}^{1} \frac{d\xi}{(1 - \xi^2)^{1/2}(\xi - x)} \int_{-1}^{1} F_n(\xi, t) g_n(t) \, dt \]
\[ -\frac{(1 - x^2)^{1/4}}{\pi} \int_{-1}^{1} \frac{f_n(\xi) d\xi}{(1 - \xi^2)^{1/2}(\xi - x)} \quad n = 1,2; \quad |x| < 1. \]  

(20)

\( F(\xi, t) \) are bounded as \( |\xi - t| \to 0 \); thus the order of integration in (20) can be interchanged (see the Poincaré transformation formula\(^{11}\)). Thus,

\[ g_n(x) = \frac{(1 - x^2)^{1/4}}{\pi} \int_{-1}^{1} G_n(x, t) g_n(t) \, dt + \frac{(1 - x^2)^{1/4}}{\pi} h_n(x) \quad |x| < 1; \quad n = 1,2 \]  

(21)

where

\[ G_n(x, t) = \int_{-1}^{1} \frac{F_n(\xi, t) d\xi}{(1 - \xi^2)^{1/2}(\xi - x)} \]  

(22)

\[ h_n(x) = -\int_{-1}^{1} \frac{f_n(t) d\xi}{(1 - \xi^2)^{1/2}(\xi - x)}. \]  

(23)

Since \( G_n \) is of the order \( \omega^2 \) as \( \omega \to 0 \), it follows that for \( \omega \) small enough, a solution to (21) can be obtained by successive approximations. If terms of order \( \omega^2 \) are neglected, then a first approximation is given by

\[ g_n(x) = \pi^{-1} (1 - x^2)^{1/4} h_n(x) \quad n = 1,2, \quad |x| < 1. \]  

(24)

Since we can write the identity

\[ \int_{-1}^{1} \frac{d\xi}{(1 - \xi^2)^{1/2}(\xi - x)} = 0 \quad |x| < 1, \]
$h_n(x)$ can be computed to within terms of order $\omega^1$. Thus,

$$g_1(x) = \frac{2\alpha^2 - k_2^2}{k_1^2 - k_2^2} (1 - x^2)^{1/4}, \quad |x| < 1 \tag{25}$$

$$g_2(x) = \frac{2\alpha \beta}{k_1^2 - k_2^2} (1 - x^2)^{1/4}, \quad |x| < 1. \tag{26}$$

Since

$$\int_{-1}^{1} \frac{e^{i\eta \zeta}}{(1 - \zeta^2)^{1/2}} d\zeta = \pi J_0(\zeta),$$

we have

$$(\zeta^2 - k_1^2)^{1/4} R_1(\xi) = \frac{\alpha^2 - \frac{1}{2} k_2^2}{k_1^2 - k_2^2} J_1(\xi)$$

$$(\zeta^2 - k_2^2)^{1/4} R_2(\xi) = \frac{\alpha \beta}{k_1^2 - k_2^2} J_1(\xi). \tag{27}$$

The stresses in the immediate neighborhood of the edges of the crack $y = 0$, $|x| = 1$ depend on the behavior of the integrands for $|\xi|$ large. In view of the behavior of $J_1(\xi)$ for $|\xi| \gg 1$, the stresses are of order $r^{-1/4}$ as $r \to 0$, where $r$ is the distance from either edge of the crack.

To find the far field we combine (2), (6), (9), and (27), and evaluate the resultant integrals by the method of steepest descents. In polar coordinates $x = R \sin \theta$, $y = R \cos \theta$, we get

$$U_R =$$

$$\frac{k_1^2}{k_1^2 - k_2^2} \left( \frac{2\pi k_1}{R} \right)^{1/4} \left[ \sin^2 \theta - \frac{k_2^2}{2k_1^2} \left( \alpha^2 - \frac{1}{2} k_2^2 \right) - \alpha \beta \sin \theta \cos \theta \right] e^{i(kR - \pi/4)}$$

$$U_\theta =$$

$$\frac{k_2^2}{k_1^2 - k_2^2} \left( \frac{2\pi k_2}{R} \right)^{1/4} \left[ \sin \theta \cos \theta \left( \alpha^2 - \frac{1}{2} k_2^2 \right) + \alpha \beta (\sin^2 \theta - \frac{1}{2}) \right] e^{i(kR - \pi/4)}$$

valid at large distances and to terms of the lowest order in $k$.

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A THEOREM ON THE ISOTROPY GROUPS OF A HYPERELASTIC MATERIAL*

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If an elastic material is hyperelastic, its symmetries at a given particle are specified in terms of two sets of groups: the isotropy groups \( g \) of its response functions, for all reference configurations, and the isotropy groups \( g_e \) of its stored-energy functions, \( \sigma \). For a particular reference configuration, the group \( g \) corresponds to the set of deformations of the material that cannot be detected by measurements of contact forces, \( g_e \), to those that cannot be detected by measurements of elastic energy. It is widely believed, if often tacitly, that these two classes of deformations are the same. The following theorem shows the common opinion to be true for solids and fluids but false in general. Moreover, in an isotropic hyperelastic material, \( \sigma \) is shown to be an isotropic function. In the statement of the result, the reference configuration is taken as the same for \( g \) and for \( g_e \), though otherwise arbitrary.

Let \( U = \{ H \} \) and \( O = \{ Q \} \) be the unimodular and full orthogonal groups, respectively; let \( F \) be an arbitrary tensor; let \( S \) stand for a positive-definite, symmetric tensor; let \( I \) be the identity tensor.

**Theorem.** In a hyperelastic material

\[
\mathfrak{g}_e \subset \mathfrak{g},
\]

\[
H \in \mathfrak{g} \leftrightarrow VF: \quad \sigma(F) = \sigma(FH) + \sigma(I) - \sigma(H).
\]

**Corollary 1.**

\[
Q \in \mathfrak{g} \rightarrow Q \in \mathfrak{g}_e.
\]

Thus, by (1), the orthogonal subsets of \( \mathfrak{g} \) and \( \mathfrak{g}_e \) are identical:

\[
\mathfrak{g} \cap O = \mathfrak{g}_e \cap O.
\]

**Corollary 2.** In every hyperelastic fluid or solid, \( \mathfrak{g} = \mathfrak{g}_e \).