SINGULARITIES IN SPATIALLY HOMOGENEOUS, DUST-FILLED
COSMOLOGICAL MODELS

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Most cosmological models in general relativity have a point of singularity: There is a point, which can be reached by a geodesic of finite total length from other points of the space-time manifold, where the metric is degenerate or otherwise irregular (for example, a point where a curvature scalar is infinite). All known dust-filled models, without cosmological constant, have singular points, and it may be conjectured that the presence of dustlike matter filling space always leads to a singularity. This paper shows that this conjecture is true for an important class of universes.

Some cosmological models, not of the dust-filled type, do not have this geometric imperfection. There exists a vacuum solution, the Taub-NUT-Misner\textsuperscript{2-4} universe (here denoted T-NUT-M), which is a cosmological model in that it admits closed spacelike homogeneous hypersurfaces. This universe, moreover, does not have a point of singularity in the sense described above. This model, however, is unusable as a cosmological model because of incompleteness: Some timelike geodesics are of finite total length and yet do not have any limiting end point.

The T-NUT-M solution is “symmetric” under the three-dimensional Lie group of Bianchi type nine [denoted SO(3,R)]. A symmetry group is a group of transformations of the space-time manifold: Under the action of the group elements, any single given point of the manifold traces out a subspace—in the cases under consideration, a three-dimensional hypersurface—at all of whose points the metric tensor is the same. Some of these invariant hypersurfaces are spacelike, but not all. A manifold with this type of symmetry is said to be “spatially homogeneous.”

In a search for information concerning the development of singularities in realistic cosmological models, one is led to impose the same symmetry conditions. Unlike the T-NUT-M vacuum solution, however, every dust-filled universe of this kind possesses a singularity; this paper will sketch the proof of this fact.

Why is it desirable to look at spatially homogeneous models? Why use the group SO(3,R) instead of another? There are four reasons:

First, the introduction of any symmetry group enormously simplifies the problem of solving the field equations. (The danger, of course, is that any such simplification is too much!)

Second, this particular group stands out—it allows the introduction of three directions which are equivalent as far as group properties are concerned: The structure constants of the group SO(3,R) assume a natural form when expressed in a characteristic basis of the tangent space of the group at the unit element of the group (the group algebra). The structure constants remain unchanged when this basis, which is three-dimensional, is transformed by means of a three-dimensional orthogonal transformation of determinant one (a rotation). This means that although the directions within the three-dimensional invariant hypersurfaces may not be equivalent metrically, they are similar in their group properties, which are
expressed in terms of commutation relations between contravariant vectors. These directions are, because of this, equivalent topologically. The invariant hypersurfaces all have the topology of a three-sphere so that all directions are finite ones.

The similarity of spacelike directions in the real world makes it natural to require that at least one of these invariant hypersurfaces be spacelike: A cosmological model would indeed be improper unless it contained a region approximating the observed vicinity of the cosmos. The topology and group conformities granted to a spacelike invariant hypersurface by SO(3,R) offer at least a minimal response to a request for such a portrait of the physical world. 6

Another three-dimensional symmetry group gives the same topology and group symmetry within each invariant hypersurface. It is the group of Bianchi type one (called T3). In this case geodesics in all directions in an invariant surface are infinite. The models with this symmetry are all known to possess singular points.1 Thus the symmetry group of type nine, SO(3,R), remains at the center of attention.

Third, the topology of the invariant hypersurfaces under SO(3,R) is compact, being that of a three-sphere. Thus if the dust were in the form of galaxies, only a finite number would exist, and no apparent paradoxes due to infinite numbers of stars would be encountered. On the other hand, some investigators argue that a finite universe is convenient but far from obligatory. This third reason, then, merely suggests that SO(3,R) is a satisfying choice; this point does not decide the issue of which group to choose.

Fourth, the introduction of SO(3,R) still allows great generality in the possible solutions to the field equations. In particular, there exist solutions where the rotation, in the sense of Raychaudhuri,7 is arbitrarily large. (The symmetry group T3 does not allow rotation.) This is important since it is relatively simple to show that space-times with no rotation have a singularity; the question is whether members of a general class of spaces, where rotation is possible, need have singular points. This paper answers that they do!

(Wright8 has exhibited a dust-filled model, with rotation, in which no singularity occurs along the world-lines of the dust. However, the scalar curvature becomes infinite at a finite spatial distance from an axis of symmetry. His universe has neither the equivalence between spacelike directions nor the closed space topology of the present models.)

This paper, then, is concerned with cosmological models having the special but reasonable three-dimensional symmetry of SO(3,R). At every point in the invariant hypersurfaces the matter density, matter velocity, and metric are all the same; isotropy, however, is not assumed: At any given point the metrical properties of different directions may be different. The geometry of at least one invariant hypersurface is spacelike.

This paper discusses only dust-filled cosmological models: On the one hand, it is insufficient to look at vacuum solutions—the incompleteness of the T-NUT-M universe makes it unusable as a cosmological model. On the other hand, nongravitational interactions, pressure terms, and equations of state are not important to the issue of existence of singularities (dust sufficiently dilute).9 It is thus understood that “cosmological model” means an exact solution of the general relativity equations with the stress-energy tensor of dustlike matter:
The function $\rho$ is the proper matter density, and the functions $u_i$ $(i = 0, 1, 2, 3)$ are the components of the matter velocity, a unit timelike vector field.

The main result is the

**Theorem.** All dust-filled cosmological models in general relativity which have the symmetry of $SO(3, R)$ (the group of Bianchi type nine) and which have at least one spacelike invariant three-dimensional hypersurface have a point of singularity.

Before sketching the lemmas involved in the proof, some comments will be made. Use is made of the notation and methods of modern differential geometry.\(^4\) The invariant hypersurfaces, which are three-dimensional, form a one-parameter family filling the four-dimensional space-time manifold, and thus can be labeled by a parameter $\tau$. They will be called $H(\tau)$ and will be said to "evolve" as the parameter $\tau$ changes.

Many different methods of putting "coordinates" on the individual $H(\tau)$ may be used. The word "coordinates" is put in quotes because the process in general results in a four-tuple system or basis. This is a set of four independent vector fields, not necessarily derived from a true coordinate system. With respect to such a basis components of tensors can be found and manipulated.

There are two four-tuple systems which are more natural than others. One, the "comoving" system, is such that the particles of dust appear to be at rest. In all cases in which nonzero rotation occurs, this system is different from the second, the "synchronous" system. In this second system, the timelike basis vector is perpendicular to the invariant hypersurfaces, and $\tau$ is a label of proper time. In either system, the four-tuple basis may be chosen so that components of the metric tensor, the matter density, and the matter velocity vector are functions of $\tau$ only, and not functions of position in an invariant hypersurface. This, of course, is to be expected since in any particular invariant hypersurface the metrical and material properties of the model are the same everywhere.

The synchronous system, however, has its severe limitations. Once a basis has been settled on, the determinant of the metric tensor, being a function of $\tau$, is said to "evolve" as $\tau$ varies. The following fact has been proved by Lifshitz and Khalatnikov:\(^{11}\)

**Lemma 1.** In the synchronous basis, the determinant of the metric tensor evolves, from any given starting point, to the value zero in a finite proper time. As long as the $H(\tau)$ remain spacelike, the synchronous system is a valid basis to use.

Consequently, Lemma 1 says that after a finite proper time of evolution either a singularity occurs or else $H(\tau)$ develops a lightlike geometry.

This latter phenomenon is exactly what appears in the vacuum T-NUT-M space.\(^4\) The $H(\tau)$ are spacelike in a finite interval of the parameter $\tau$; at either end of this interval $H(\tau)$ develops a lightlike geometry, and then becomes timelike, but no geometric singularity appears.

In contrast, a true singularity always occurs in a dust-filled universe. The proof, sketched below, shows that a lightlike geometry for an $H(\tau)$, such as occurs in T-NUT-M space, is inconsistent with the general relativity equations for dustlike matter. Since any spacelike $H(\tau)$ must evolve to this lightlike situation in a finite
proper time, but must be stopped there in a dust-filled cosmos, a singularity must develop.

The proof of the incompatibility between a lightlike \( H(\tau) \) and the stress-energy tensor of dustlike matter takes two parts. First, a suitable four-tuple basis is exhibited, one which is valid around any lightlike \( H(\tau) \), for any stress-energy tensor. Second, the Ricci tensor components are calculated in this basis. It is then shown that dustlike matter is inconsistent with the Ricci tensor at the lightlike hypersurface.

**Lemma 2.** Suppose \( H(\tau) \) are the invariant hypersurfaces in a manifold symmetrical under \( SO(3,\mathbb{R}) \). Suppose \( H(\tau_0) \) has a lightlike geometry: one lightlike and two space-like eigendirections. Then in a neighborhood of a point in \( H(\tau_0) \) there exists a basis \( \sigma^i \) (\( i = 0,1,2,3 \)) that the four curves or differential forms comprising the basis) of the differential forms (covariant vectors) such that:

(i) \( \sigma^0 \) is a gradient:

\[
\sigma^0 = dt,
\]

where \( t \) is a parameter labeling the \( H \)'s, and \( t = 0 \) is the lightlike \( H \). The operator \( d \) is the curl (or gradient) operator of differential geometry.

(ii) The \( \sigma^\mu(\mu = 1,2,3) \) are a \( t \)-dependent linear combination of three differential forms \( \omega^\mu \):

\[
\sigma^\mu = b^\mu \omega^\nu.
\]

\( B(t) = (b^\nu(t)) \) is a nonsingular matrix at each instant \( t \). \( B(t) \) is diagonal at \( t = 0 \).

(iii) The \( \omega^\mu \) obey:

\[
d\omega^\mu = \frac{1}{2}C^\mu_{\alpha\beta}\omega^\alpha \wedge \omega^\beta,
\]

where the \( C^\mu_{\alpha\beta} \) are the structure constants of the group (Bianchi type nine, \( SO(3,\mathbb{R}) \)): the structure constants are \( C^\mu_{\alpha\beta} = \epsilon_{\mu\alpha\beta} \). The “product” denoted by \( \wedge \) is the wedge product of differential geometry.

(iv) In the \( \sigma^i \) system, the components of the metric tensor are:

\[
g_{ij} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & g & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

where \( g = g(t) \) is a function which is zero at \( t = 0 \).

Written differently:

\[
ds^2 = (\sigma^0 \otimes \sigma^1 + \sigma^1 \otimes \sigma^0) + g(t)\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3,
\]

where \( \otimes \) is the tensor product. The \( \sigma^i \) system is not unique since the label \( t \) can be chosen quite arbitrarily.

The idea of the proof is this: There exist three contravariant vector fields \( \xi^\mu(\mu = 1,2,3) \), called the Killing vectors. These vector fields generate directions along which the metric is constant. Moreover, they correspond to the basis of the group algebra in which the group structure constants take the form \( C^\mu_{\alpha\beta} \). Three contravariant vector fields \( Y_\mu \) can be found which lie in the invariant hypersurfaces and which are invariant under Lie differentiation by the \( \xi^\mu \). For any given labeling...
\( \tau \) of the invariant hypersurfaces, these \( Y_\mu \) may be chosen to satisfy the commutation relations:

\[
[Y_\mu, \ Y_\nu] = -\epsilon^{\alpha \mu \nu} Y_\alpha.
\]  

(7)

These \( Y_\mu \) are unique only up to an orthogonal transformation (constant in \( \tau \). The metric of the three-dimensional hypersurfaces expressed in terms of the \( Y_\mu \) is called \( H = (h_{\mu \nu}) \). The arbitrariness of the \( Y_\mu \) can be used to diagonalize \( H \) at any given \( \tau \). This is done at \( \tau = t_0 \).

Now with \( t = \tau - t_0 \), a natural covariant vector is the differential form \( dt \), the gradient of \( t \). There are thus available three contravariant vectors \( Y_\mu \) and a covariant vector \( dt \). From these vectors is obtained a set of four contravariant vectors and their duals, four covariant vectors, which produce the desired form of the metric tensor components.

This is done in the following way: The eigendirections of \( h_{\mu \nu} \) define linear combinations of the \( Y_\mu \) with respect to which the metric takes on a diagonal form. Since two eigenvalues of \( h_{\mu \nu} \) are positive near \( t = 0 \), the corresponding two linear combinations, \( X_1 \) and \( X_3 \), can be normalized to unit length. The third eigendirection \( X_2 \) has length \( \beta (t) \), where \( \beta (0) = 0 \). The matrix \( (h_{\mu \nu}) \) is diagonal at \( t = 0 \); accordingly, the matrix representing the linear transformation between the \( Y_\mu \) and \( X_1, X_2, X_3 \) is diagonal at \( t = 0 \). Moreover, this matrix is nonsingular in a neighborhood of \( t = 0 \).

The four-dimensional manifold requires four contravariant basis vectors in each tangent space. To complete the three already obtained, a fourth, \( \hat{X}_0 \), is chosen. The metric components with respect to four basis vectors form a nonsingular matrix. Consequently, at \( t = 0 \) the \( (01) \) component of the metric must be nonzero. It can be presumed that \( \hat{X}_0 \) is chosen so that it is lightlike and so that the \( (02) \) and \( (03) \) metric components are zero.

The contractions \( (\hat{X}_1, dt), (X_3, dt), \) and \( (X_3, dt) \), which are contractions between contravariant vectors and a covariant one, are all zero since \( t \) is constant on each invariant hypersurface. Hence \( (\hat{X}_0, dt) \) is nonzero, and by the group property is a function of \( t \) only. Set \( \hat{X}_0 = \hat{X}_0 / (\hat{X}_0, dt) \).

In terms of \( X_0, \hat{X}_1, X_2, X_3 \), the \( (01) \) component of the metric, a function of \( t \) only, is still nonzero in a neighborhood of \( t = 0 \). Calling this component \( \hat{g}_{01} \), set \( X_1 = \hat{X}_1 / \hat{g}_{01} \). With respect to \( X_0, X_1, X_2, X_3 \) the metric now has the form of equation (5). In all the above manipulation, \( \hat{g}(t) \) has changed to \( g(t) \), but is still zero at \( t = 0 \).

The four \( X_t \) and four \( Y_t \), where \( Y_0 = X_0 \), form two bases of the contravariant vector fields. Taking duals generates \( \sigma^t \) and \( \omega^t \), respectively. The construction method guarantees that \( \sigma^0 = \omega^0 = dt \), and the other relations in the statement of the lemma may be easily verified.

**Lemma 3.** When \( H(\tau) \) is lightlike, the Ricci tensor, in a space with the properties listed in Lemma 2, is incompatible with the stress-energy tensor of dustlike matter.

This is a computational result, using components of tensors expressed in the basis found by Lemma 2. Much work is saved by the following observations: There are six independent functions among the \( b^{\alpha \gamma} (t) \) and \( g(t) \) of Lemma 2; the six Einstein equations involving second derivatives of these functions can be used to find the past and future development of any consistent set of initial data. Therefore, any contradiction must occur among the four remaining equations, which take the form
of constraints in initial data. It turns out to be enough to consider one of these equations, the (11) component of equation (1).

The only combination of the (00), (01), (11) components of equation (1) which does not involve second derivatives of the functions $b^a$ $(l)$ and $q(l)$ yields, at $t = 0$, when $g = 0$:

$$R_{11} = \rho u_1^2 = a \text{nonnegative quantity.}$$  \hspace{1cm} (8)

On the other hand, an explicit calculation of $R_{11}$ at $t = 0$ gives

$$R_{11} = -\alpha^2,$$  \hspace{1cm} (9)

where $\alpha^2$ is positive or at best zero.

That $R_{11}$ in equation (8) is positive, and not merely nonnegative, follows from the "conservation law," $(\rho u^a u^l)_{;l} = 0$. Thus, if $\rho u^0 \text{det}(B) \neq 0$ anywhere (that is anywhere in the region in which the basis of Lemma 2 is valid), then $\rho u^0 \text{det}(B) \neq 0$ at $t = 0$. Hence $\rho$ and $u^0$ must be nonzero at $t = 0$. Moreover at $t = 0$, $u_1 = u^0$. Therefore $\rho u_1^2$ is positive at $t = 0$. Consequently, equations (8) and (9) are inconsistent.

This completes the sketch of the proof of the theorem. A singularity occurs in every cosmological model considered here!

Since the singularity happens when $H(\tau)$ becomes lightlike, the synchronous system may be used over the entire nonsingular range of any of the present models. In this basis the conservation law $(\rho u^a u^l)_{;l} = 0$ can be explicitly integrated to give

$$\rho u^0 G^{1/2} = \text{constant},$$  \hspace{1cm} (10)

where $G$ is the determinant of the metric.

When $G$ becomes zero, and the space becomes singular, $\rho$ may become infinite. This happens in the Friedmann model; in general whenever there is no rotation (for then $u^0 = 1$, constant), the universe ends in this fashion. It may happen, however, that $u^0$ becomes infinite as $G$ becomes zero, and that $\rho$ stays finite. The matter in the universe would not approach a collapsed state of infinite density.

Thus the behavior of $\rho$ does not necessarily provide a good symptom that a singularity is drawing near. The question of what happens as the singular point approaches is presently under investigation.

Computer calculations have exhibited some interesting specific models. Included in the cases computed are some with very high rotation. All, however, are compatible with either of the two possibilities: (1) $\rho$ becomes infinite at the singularity in all universes with or without rotation; or (2) $\rho$ stays finite in all universes with nonzero rotation.

One final comment can be made: The synchronous system can be used for spaces symmetric under the group of Bianchi type one, $T_3$, where $C^a_{ab} = 0$. The solution of the conservation law equation is of the same form as equation (10). But rotation is always zero in this situation: $u^0 = 1$, constant. Therefore the singularity is necessarily of the type $\rho$ becoming infinite; this fact is known.\footnote{1}

The complete proofs of the various points only sketched here, a detailed discussion of the calculations and the methods involved, and the numerical computations will be presented in a paper now in preparation.
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7 Raychaudhuri, A., Phys. Rev., 98, 1123 (1955); also see ref. 1.
11 The lemma, stated more fully, says that either the determinant of the metric is constant or else it goes to zero in a finite proper time; the option of a constant determinant is not allowed in a dust-filled space. The approach to zero determinant may be either toward the past or toward the future, and not necessarily in both directions. This lemma is proved by Lifshitz and Khalatnikov (ref. 5), who remark that this fact was originally pointed out by L. D. Landau (no ref. given). This lemma was proved by A. Komar, Phys. Rev., 104, 544 (1956), and in the special case of zero rotation by Raychaudhuri (ref. 7).

DISTAL FUNCTIONS

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Let $G$ be a fixed abelian topological group. A pair $(X,G)$ is a flow if $X$ is a compact Hausdorff space and if $G$ acts on $X$ in a jointly continuous manner. The flow is minimal if every orbit is dense. It is distal if whenever $\{g_n\}$ is a net (i.e., a generalized sequence indexed by any directed set) in $G$ with $g_n x \to z$ and $g_n y \to z$, then $x = y$. Auslander and Hahn introduced classes of distal and minimal functions defined on $G$. A distal function is any complex-valued function defined on $G$ which is equivalent to the restriction to a single orbit of a continuous function defined on the space $X$ of some distal flow $(X,G)$, and a minimal function on $G$ arises in this manner from a minimal flow. Among other things, they proved that the distal functions on $G$ form a Banach algebra (with the supremum norm), but, for instance, the minimal functions on the additive group of reals are not closed under addition. Earlier work of Ellis implied that almost periodic functions are distal and that distal functions are minimal.

The purpose of the present note is to announce two structure theorems for