\[
\frac{\partial (q^1, \ldots, q^n)}{\partial (t^1, \ldots, t^n)}
\]  

(11)

is a density, and so is its reciprocal. Therefore the reciprocal exists as a holomorphic function, which means that on the natural completion the continuations of the functions (10) continue to be locally functionally independent. Therefore the base space of the natural completion can be spread out "regularly" over \(C^n\), and is thus a part of \(\bar{D}\), which proves the theorem.

**Remark 1:** If \(D\) itself is many-sheeted to begin with, then the coordinates (10) guarantee at any rate part (ii) of Definition 1. If now there are enough holomorphic functions on \(D\) to supply part (i), then Theorem 3 remains in force.

**Remark 2:** If, however, there are not enough holomorphic functions on \(D\) to do that, then we may first create a new domain \(D^o\) by identifying points of \(D\) in which all holomorphic functions, including (10), have the same functional elements, and now* Theorem 3 applies to \(D^o\).

**Remark 3:** The original Weierstrass continuation of a holomorphic function \(f\), say in a domain \(D\) of \(C^n\), becomes a natural completion if we introduce the system (1), in which \(S = D\), and \(\{f_m\}\) consists not only of the given function \(f\) but also of the functions (10), and of the Jacobian (11) and its reciprocal. It is thus highly "nonintrinsic" as far as the function \(f\) itself is concerned.

**Remark 4:** Since meromorphic functions, as generalizations of holomorphic ones, are also subject to Lemma 1. Theorem 3 also applies, literally, to envelopes of meromorphy.

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4 This is the so-called "Zorn's lemma." It began to be thus designated in the mid-1930's, but it had been formulated, very expressively, in our ref. 1.


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**ON THE IDENTIFICATION OF SYSTEMS AND THE UNSCRAMBLING OF DATA, I. HIDDEN PERIODICITIES**

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**Introduction.**—The greater part of classical and contemporary mathematical analysis is devoted to the problem of predicting the future behavior of a system, given the defining equations and the initial state. This is rather surprising since the basic scientific problem, in such fields as physics, engineering, biology, medicine, and economics, to mention only a few, is that of determining the structure of the
system, given various observations over time. Actually, the problem is a deeper one of determining the best approximate system that agrees sufficiently well with the available data. Furthermore, what is “best” involves a consideration of the intended use of the model.

These identification or “inverse” problems are of a high degree of mathematical complexity (cf. the papers by Borg,¹ and Gelfand and Levitan²). The numerical aspects are, however, of an even higher degree of complexity. Only within the last 10 years, with the emergence of the fast computer with a reasonably large rapid access storage (“memory”), have some of these inverse problems become tractable. In this series of papers, we wish to outline the application of new analytic techniques such as quasilinearization, differential approximation, and dynamic programming to the computational solution of a variety of identification and unscrambling problems. Detailed discussions will be found in subsequent papers; for an application to cardiology, see reference 3.

An Unscrambling Problem.—A basic fact of experimental life is that measuring devices “hear” many different signals from many different sources. A fundamental problem, then, is that of determining what source provided what stimulus. The classical periodogram analysis problem is of this genre (cf. Whittaker⁴). Given the values $f(t_i)$ where $f(t) = \sum_{i=1}^{R} \alpha_i \cos \omega_i t_i$, we wish to determine the parameters $\alpha_i$ and $\omega_i$. Another problem of frequent occurrence is that of determining the $\lambda_i$, given the values $f(t_i)$, where $f(t)$ now has the form $f(t) = \sum_{i=1}^{R} \beta_i e^{\lambda_i t_i}$. These problems are notoriously difficult (cf. Lanczos⁵).

As we have pointed out before (cf. refs 6 and 7), these problems are special cases of that of determining the coefficients $a_i$ and the initial values $c_i$ when we know that $f(t)$ satisfies the linear differential equation $f^{(N)} = \sum_{i=0}^{N-1} a_i f^{(i)}(t_i) = c_i$, $i = 0, 1, 2, \ldots, N - 1$. This, in turn, is a special case of the problem of determining the vectors $a'$ and the initial vectors $c'$ when $f(t)$ is known to have the form $f(t) = \sum_{i=1}^{N} u_i(t, a')$, where the functions $u_i(t, a')$ are solutions of nonlinear differential equations of the form $\dot{u}_i = g_i(u, a')$, $u(0) = c'$. We call this the problem of “differential approximation.” Particular versions of this problem, in the case where the defining equations are linear, have been treated by Medgnyess,⁶ using a quite different approach from the one which we shall employ. References to biomedical and biochemical applications may be found there.

The Approach.—Suppose that the $N$-dimensional vector $X(t)$ is a solution of the differential equation $\dot{X} = f(X, a)$, $X(0) = c$, where the system parameter vector $a$ and the initial state vector $c$ are as yet unspecified. At certain times the system is “observed,” which results in the approximate conditions $\beta_i, X(t_i) \cong b_i$, $i = 1, 2, \ldots, M \geq N$. We wish to determine the vectors $a$ and $c$ so that we minimize $S$, the sum of the squares of the deviations $S = \sum_{i=1}^{M} (\beta_i, X(t_i) - b_i)^2$. If we consider the vector $a$ to be a variable, but subject to the condition $\dot{a} = 0$, then we can adjoin this equation to the original equation. Now our problem is to determine the initial conditions $c$ and $a(0)$ so as to minimize the $S$. 
Computational Formalism.—We wish to minimize a sum \( S = \sum_{i=1}^{M} |(\beta_i, X(t_i)) - b_i|^2 \), where \( \dot{X} = f(X) \), and the minimization is over the initial vector \( c, X(0) = c \). Our procedure is an iterative one. First we select an initial approximation \( c^0 \). In addition we integrate the differential equation \( \dot{X} = f(X^0), X^0(0) = c^0 \), numerically on the interval \( 0 \leq t \leq t_M \). Next we consider a linearized version of the foregoing equation \( \dot{X} = f(X^0) + J(X^0)(X^1 - X^0) \), where the elements in the matrix \( J \) are \( J_{ij} = \frac{\partial f_i(X^0)}{\partial X_j} \). These elements are known computationally for \( 0 \leq t \leq t_M \). We produce numerically a particular solution \( p(t) \) of the equation \( \dot{p} = f(X^0) + J(X^0)(p^1 - X^0) \), \( p(0) = 0 \), and then \( N \) independent solutions of the homogeneous equation, \( h_1(t), h_2(t), \ldots, h_N(t), h_i = J(X^0)(h_i - X^0) \), where the \( j \)th component of \( h_i(0) = \delta_{ij} \). Then the function \( X^1(t) \) is representable in the form

\[ X^1(t) = p(t) + \sum_{i=1}^{N} c_i h_i(t). \]

The constants are determined as the solution of the linear algebraic equations \( \partial S/\partial c_i = 0 \), where for \( X(t_i) \) we substitute \( X(t_i) = p(t_i) + \sum_{j=1}^{N} c_j h_j(t_i), i = 1, 2, \ldots, M \). These values of \( c_1, c_2, \ldots, c_N \) we take to be our new approximations of the optimal initial conditions. Then the process is repeated.

Let us call attention to the fact that we are assuming that the particular and homogeneous solutions can be computed accurately on the interval \( 0 \leq t \leq t_M \) and that the indicated system of linear algebraic equations can be solved computationally. In some instances these assumptions are not met, though we have experienced little difficulty in the examples which we have treated. Furthermore, we are assuming that the initial approximation is sufficiently good so that the sequence of approximations converges. When convergent, the rate is very fast, since the algorithm is quadratic in nature; we approximately double the number of correct digits with each additional step.

Some Examples.—Consider the first problem mentioned in the Introduction. The function \( f(t) \) has the form \( f(t) = \sum_{i=1}^{R} \alpha_i \cos \omega_i t \). We put \( u_i(t) = \alpha_i \cos \omega_i t \) and observe that \( \ddot{u}_i + \omega_i^2 u_i = 0, u_i(0) = d_i, \dot{u}_i(0) = 0 \). We let \( \alpha_1 = 1.0, \alpha_2 = 0.5, \omega_2 = 0.1, \omega_1 = 1.1, \omega_3 = 2.03, \omega_4 = 3.42 \), and calculated the values of \( f(0), f(1), \ldots, f(5) \). Then using these as the observed data, we tried to use the method sketched above to estimate \( \alpha_1, \alpha_2, \ldots, \omega_4 \). With errors of from 10 to 100 per cent in the initial estimates, we almost succeeded. The estimates of \( \omega_3 \) were not particularly close; when we used more observations at times 0, 5/6, 10/6, \ldots, 30/6, all the unknowns were accurately obtained.

Consider next a linear second-order system subject to an unknown input \( r(t) \), \( \ddot{u} + 3 \dot{u} + 2u = r(t) \), where the input function \( r(t) \) is subject to the equation \( \ddot{r} + cr + dr = 0 \). Observations are made on the function \( u(t) \), and we wish to deduce the nature of the input \( r(t) \). We considered the case for which \( u(0) = 0, r(0) = 0, c = 0, \dot{u}(0) = 1, \ddot{r}(0) = 4\pi, d = (4\pi)^2 \). Values of \( u \) were produced at times \( t = 0, 0.1, \ldots, 1.0 \). Then using these 11 values of \( u \) and the initial estimates \( u^0(0) = 0.5, r^0(0) = 0.5, c = 1, \dot{u}^0(0) = 0.5, \ddot{r}^0(0) = 0.5, d = 100 \), one application of the method gave estimates that were accurate to three figures. Notice that our procedure allows us to approximate to \( r(t) \) in terms of exponential, polynomial, or trigonometric functions, depending on the values of the estimates of \( c \) and \( d \) which
are obtained. Other experiments showed that as the accuracy of the observations is lessened, the accuracy of the estimates falls. Useful information about the accuracy required in the observations to obtain estimates of prescribed accuracy is thus easily obtained by means of numerical experiments.

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ON THE IDENTIFICATION OF SYSTEMS AND THE UNSCRAMBLING OF DATA, II. AN INVERSE PROBLEM IN RADIATIVE TRANSFER*

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The advent of artificial satellites makes it desirable to be able to use measurements of the angular dependence of diffusely reflected sunlight from a planetary atmosphere to deduce the physical properties of the atmosphere. In this paper it is shown that a combination of invariant imbedding and quasilinearization may be useful for this purpose. Some illustrative numerical results are provided.

In the first paper in this series, we presented a general formulation of a significant class of inverse problems and outlined the use of quasilinearization as a systematic procedure for obtaining numerical solutions. In this paper we wish to indicate the application of these techniques to some important questions in radiative transfer. Our investigations are aimed at the complex problem of determining the nature of planetary atmospheres on the basis of various observations of the angular dependence of diffusely reflected light, and are pertinent to the design of experiments in this area. The methods, however, are applicable to a wide class of physical processes.

Statement of Problem.—Consider an inhomogeneous, plane-parallel, nonemitting, and isotropically scattering atmosphere of finite optical thickness $\tau$, whose optical properties depend only upon $\tau$, the optical height above the bottom (see Fig. 1).