\[ \text{Im} \left( sz + t(s)(Mz') \right) \geq s \text{Im} \ z_1 - \left| t(s) \right| \left| (Mz') \right| \geq s \text{Im} \ z_1 - C^{'s} \left( \frac{1}{p} \right) \left| (Mz') \right| \geq \alpha s \]

for some \( \alpha > 0 \). Therefore the integral converges and \( F(z, z') \) is an analytic function of \( z \), for \( \text{Im} \ z_1 > 0 \). Also \( F(z, 0) = -1/(2 \pi \text{Im} z) \). Let \( \chi \) denote the characteristic function of \( \{ z : \text{Im} \ z_1 > 0 \} \). Let \( u = [\chi F] \). Then \( u \in B(\mathbb{R}^n) \) and \( P(D)u = 0 \). Also, \( u \) is real analytic in \( z_1, \ldots, z_n \), but \( u(x_1, 0, \ldots, 0) \not\in \mathcal{L}_{L^1}^{\text{loc}} (\mathbb{R}) \).

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\(^8\) Professor Bengel has independently proved that \( (a) \) and \( (b) \) are equivalent (personal communication).

\(^9\) Professor Komatsu has a proof in the variable coefficient case based on an estimate in his paper "A proof of Katake and Narasimhan’s theorem," \textit{Proc. Japan Acad.}, 38, 615 (1962).


THE PLEMELJ FORMULAS WITH UNRESTRICTED APPROACH, AND THE CONTINUITY OF CAUCHY-TYPE INTEGRALS*

BY FREDERICK BAGEMIHL

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MILWAUKEE

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Let \( J \) be a Jordan arc or a Jordan curve, and suppose that \( J \) is oriented and rectifiable. Assume that the function \( f(\zeta) \) is defined and summable on \( J \). Then the Cauchy-type integral

\[ \int_J \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad (z \not\in J) \tag{1} \]

represents a holomorphic function in each region complementary to \( J \). If \( \zeta_0 \in J \) but \( \zeta_0 \) is not an end point of \( J \), then the singular Cauchy integral

\[ \int_J \frac{f(\zeta)}{\zeta - \zeta_0} \, d\zeta \quad (\zeta_0 \in J) \tag{2} \]
is said to exist at \( \zeta_0 \), provided that the integral (2) exists as a Cauchy principal value.

Now suppose that \( \zeta \in J \) but that \( \zeta \) is not an end point of \( J \). By a left arc at \( \zeta \) we mean a Jordan arc extending from a point in the complement of \( J \) to the point \( \zeta \) and lying, except for the point \( \zeta \), in the complement of \( J \), such that all points of the arc in question lying in a sufficiently small neighborhood of \( \zeta \) lie to the left of an observer traversing \( J \) in the sense of its orientation. A right arc at \( \zeta \) is defined analogously.

Let \( E^l \) and \( E^r \) be measurable subsets of \( J \), and suppose that at every point \( \zeta' \in E^l \) there is a left arc \( A_{l,\zeta'} \), and at every point \( \zeta'' \in E^r \) there is a right arc \( A_{r,\zeta''} \), with the property that the following limits of the Cauchy-type integral (1) exist:

\[
\lim_{\varepsilon \to 0^-} \int_{\zeta' - \varepsilon}^{\zeta'} f(\xi) d\xi = \lambda_{l,\zeta'}, \quad \lim_{\varepsilon \to 0^+} \int_{\zeta'' + \varepsilon}^{\zeta''} f(\xi) d\xi = \lambda_{r,\zeta''}.
\]

**Theorem 1.** If the singular Cauchy integral (2) exists almost everywhere on \( J \), then at almost every point \( \zeta' \in E^l \),

\[
\lambda_{l,\zeta'} = \frac{1}{2\pi i} \int_{\zeta - \zeta'} f(\xi) d\xi + \frac{1}{2} f(\zeta'),
\]

and at almost every point \( \zeta'' \in E^r \),

\[
\lambda_{r,\zeta''} = \frac{1}{2\pi i} \int_{\zeta - \zeta''} f(\xi) d\xi - \frac{1}{2} f(\zeta'').
\]

**Proof:** According to a theorem of Privalov,\(^1\) at almost every point \( \zeta_0 \in J \), the Cauchy-type integral (1) has an angular limit \( \gamma_{l,\zeta_0} \) from the left of \( J \) and an angular limit \( \gamma_{r,\zeta_0} \) from the right of \( J \) satisfying the relations

\[
\gamma_{l,\zeta_0} = \frac{1}{2\pi i} \int_{\zeta - \zeta_0} f(\xi) d\xi + \frac{1}{2} f(\zeta_0),
\]

\[
\gamma_{r,\zeta_0} = \frac{1}{2\pi i} \int_{\zeta - \zeta_0} f(\xi) d\xi - \frac{1}{2} f(\zeta_0).
\]

By the ambiguous-point theorem,\(^2\) there are at most enumerably many points \( \zeta' \in E^l \) for which \( \gamma_{l,\zeta'} = \lambda_{l,\zeta'} \), and at most enumerably many points \( \zeta'' \in E^r \) for which \( \gamma_{r,\zeta''} = \lambda_{r,\zeta''} \). In view of this fact, a comparison of (5) and (6) with (3) and (4) yields the conclusion of the theorem.

If, in particular, we assume that \( E^l = E^r = E \), and if we add and subtract (3) and (4), then Theorem 1 has the following

**Corollary.** If the singular Cauchy integral (2) exists almost everywhere on \( J \), then at almost every point \( \zeta_0 \in E \),

\[
\frac{1}{2\pi i} \int_{\zeta - \zeta_0} f(\xi) d\xi = \frac{\lambda_{l,\zeta_0} + \lambda_{r,\zeta_0}}{2},
\]

\[
f(\zeta_0) = \frac{\lambda_{l,\zeta_0} - \lambda_{r,\zeta_0}}{2}.
\]
Finally, let $C$ be the positively oriented unit circle and $D$ be the open unit disk. Suppose that $S$ is a subset of $C$ of measure $2\pi$, and that at every point $\zeta_0 \in S$ there is an arc $A_{\zeta}$ in $D$ along which the Cauchy-type integral

$$F(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}, \quad (z \in D),$$

where $f(\zeta)$ is defined and summable on $C$, tends to a limit $\varphi(\zeta_0)$.

**Theorem 2.** If the values of the function $\varphi(\zeta) (\zeta \in S)$ coincide almost everywhere on $C$ with the values of a function that is continuous on $C$, then $F(z)$ is continuous on $C \cup D$.

**Proof:** Because of the particular form of $C$, the function $F(z)$ has an angular limit at almost every point of $C$; and hence at almost every point of $S$. It follows again, from the ambiguous-point theorem, that at almost every point $\zeta_0 \in S$, this angular limit has the value $\varphi(\zeta_0)$. This means that the angular limits of $F(z)$ coincide almost everywhere on $C$ with the values of a function that is continuous on $C$. It is known that this implies the continuity of $F(z)$ on $C \cup D$.

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2 Bagemihl, F., "Curvilinear cluster sets of arbitrary functions," these PROCEEDINGS, 41, 379–382 (1955), Corollary 1, preceded by a conformal mapping of a region complementary to $J$ onto the unit disk.
4 Ibid., p. 139.

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**FRACTURE FACES OF FROZEN MEMBRANES**

*BY DANIEL BRANTON*

DEPARTMENT OF BOTANY, UNIVERSITY OF CALIFORNIA, BERKELEY

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The biological membrane, according to one widely accepted concept, has as its framework a bimolecular leaflet which under appropriate conditions can be seen in the electron microscope as two dark 20-Å-thick layers separated by a lighter 35-Å-thick layer.1 Well-known theories and evidence2–3 suggest that this structure is composed of a bimolecular leaflet of oriented lipid molecules sandwiched between two layers of protein. Though Robertson10 has formalized these ideas as the basis of his generalized unit membrane concept, new chemical11–13 and structural14–16 evidence requires that other molecular arrangements also be considered. This has been the case in several recently proposed membrane models. Though some of these models take as their starting point the general notion of a bimolecular leaflet17,18 and others take as a starting point a repeating particulate subunit,11,14,16 they all emphasize the possibility of dynamic interrelations between the several membrane components and explicitly deny the notion of a biological membrane which is spatially and temporally uniform.