ON THE UNIFICATION OF THE CALCULUS OF VARIATIONS
AND THE THEORY OF MONOTONE NONLINEAR OPERATORS
IN BANACH SPACES*

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The recently developed theory$^1$ of monotone nonlinear operators $T$ from a Banach space $X$ to its conjugate space $X^*$ can be considered most naturally as an extension to nonvariational problems of the basic ideas of the direct method of the calculus of variations. First of all, in practice, its simplest and most basic application is to a class of nonlinear elliptic boundary value problems$^2$ which is an extension of the class of Euler-Lagrange equations of multiple integral problems of general order, paralleling the application of Hilbert space methods for general linear elliptic problems as an extension of the variational method for self-adjoint problems. Second, on a more basic level of principle, if the operator $T$ is the derivative (Gateaux or Fréchet) of a real-valued function $f$ on $X$, the condition that $T$ be monotone is equivalent to the convexity of $f$,$^3$ the basic property (with its modifications) on which one rests the direct method of the calculus of variations in Banach spaces.$^4$ For this subclass of monotone operators $T$, the existence of solutions of the equation $Tu = 0$ is equivalent to the existence of critical points for the corresponding function $f$. On the other hand, for $f$ just semiconvex, the direct method of the calculus of variations yields not merely critical points of $f$ but extreme points of $f$, so that a certain amount of information is lost in passing from the study of variational problems to the theory of monotone operators. A similar relation holds between variational problems on convex sets and monotone operator inequalities on convex sets.$^5$

It is our object in the present note to give a unified extension of the theory of monotone operator equations, the calculus of variations, and the theory of monotone operator inequalities on convex sets. The formulation of this extension is as follows:

Let $X$ be a reflexive Banach space, $X^*$ its conjugate space with the pairing between $w$ in $X^*$ and $u$ in $X$ denoted by $(w, u)$. If $T$ is a mapping from $X$ to $X^*$ (or has its domain $D(T)$ in $X$), $T$ is said to be monotone if

$$(Tu - Tv, u - v) \geq 0$$

for all $u$ and $v$ in $D(T)$. $T$ is said to be hemicontinuous if its domain $D(T)$ is convex and $T$ is continuous from each line segment in $D(T)$ to the weak topology of $X^*$. A function $f$ from $X$ to $(-\infty, +\infty]$ (i.e., with $+\infty$ as a permitted value) is said to be convex if for each pair $u$ and $v$ of $X$ and any $\alpha$ with $0 \leq \alpha \leq 1$,

$$f(\alpha u + (1 - \alpha)v) \leq \alpha f(u) + (1 - \alpha)f(v).$$

**Theorem 1.** Let $T$ be a monotone hemicontinuous operator defined on the reflexive Banach space $X$ with values in $X^*$, $f$ a lower semicontinuous function from $X$ to $(-\infty, +\infty]$ with $f(0) = 0$. Suppose that for a given $w$ in $X^*$, there exists $R > 0$ such that

$$(Tu - w, u) + f(u) > 0$$

(1)
for all \( u \) with \( \| u \| = R \).

Then there exists \( u_0 \) in \( B_R = \{ u \| u \| \leq R \} \) such that

\[ (T_u - w, v - u_0) \geq f(u_0) - f(v), \quad v \in X. \tag{2} \]

**Theorem 2.** Under the hypotheses of Theorem 1, the set \( A_w \) of solutions \( u_0 \) of the system of inequalities (2) is a closed convex subset of \( X \). If \( T \) is strictly monotone (i.e., \( (T_u - T_v, u - v) > 0 \) for \( u \neq v; u, v \in X \)), then \( A_w \) consists of a single point.

**Theorem 3.** If in addition to the hypotheses of Theorem 1, we add the condition

\[ \{(T_u, u) + f(u)\}/\| u \| \to + \infty \quad \text{as} \quad \| u \| \to + \infty, \tag{3} \]

then a solution of the system of inequalities (2) will exist for each \( w \) in \( X^* \).

We may specialize Theorems 1–3 as follows:

(I) If \( f(u) \) is identically 0, the inequality (2) becomes the equation

\[ Tu_0 = w, \tag{4} \]

and we have the theory of monotone operator equations.¹

(II) If \( C \) is a closed convex set and \( f \) is the indicatrix function of \( C \) (i.e., \( f = 0 \) on \( C \), \( + \infty \) off \( C \)), the inequality (2) becomes the system of inequalities

\[ (T_u - w, v - u_0) \geq 0, \quad v \in C, \tag{5} \]

of the theory of monotone inequalities on convex sets.⁴

(III) If \( T = 0 \), \( f \) has a minimum at \( u_0 \) and we are in the framework of the calculus of variations.

We note that for the case of \( T \) an accretive linear operator in a Hilbert space, the results of Theorems 1, 2, and 3 have been established by C. Lescarret⁶ and Lions-Stamapchia.⁷ At the close of the present note, we give an extension of Theorem 1 to densely defined operators \( T \) which (as we shall show in detail elsewhere) suffices for the treatment of parabolic, hyperbolic, and other initial value problems involving inequalities.⁸

The proofs of Theorems 1–4 use the following lemmas:

**Lemma 1.** If \( T \) is a hemiconvex operator from \( D(T) \) in \( X \) to \( X^* \), \( f \) a convex function from \( X \) to \( (-\infty, + \infty) \) with \( f \neq \infty \), a sufficient condition that an element \( u_0 \) of \( D(T) \) be a solution of the system of inequalities

\[ (T_u - w, v - u_0) \geq f(u_0) - f(v), \quad v \in D(T), \tag{6} \]

is that \( u_0 \) be a solution of the new system of inequalities

\[ (Tv - w, v - u_0) \geq f(u_0) - f(v), \quad v \in D(T). \tag{7} \]

If \( T \) is monotone in addition, then the two systems of inequalities are equivalent.

**Proof of Lemma 1:** Suppose that the inequality (7) holds. Choosing an element \( v \) with \( f(v) < + \infty \), we see that \( f(u_0) \) is finite. Let \( x \) be any element of \( D(T) \) and for any \( t > 0 \), consider the convex combination

\[ v_t = (1 - t)u_0 + tx. \]

If we set \( v_t \) for \( v \) in the inequality (7) and use the convexity of \( f \), we find that
\[ t(Tv_t - w, x - u_0) \geq t[f(u_0) - f(x)]. \]

Cancelling \( t > 0 \), we have
\[ (Tv_t - w, x - u_0) \geq f(u_0) - f(x). \]

As \( t \to 0 \), \( Tv_t \) converges weakly to \( Tu_0 \) in \( X^* \) by the hemicontinuity of \( T \). Hence,
\[ (Tu_0 - w, x - u_0) \geq f(u_0) - f(x), \quad x \in D(T). \]

On the other hand, if \( T \) is monotone and \( u_0 \) satisfies the inequality (6),
\[ (Tv, v - u_0) \geq (Tu_0, v - u_0) \geq f(u_0) - f(v), \quad v \in D(T), \]
and the inequality (7) holds.

**q.e.d.**

**Proof of Theorem 2:** If \( f \) is convex and lower semicontinuous, for each given \( v \) in \( X \), the function of \( u \) given by
\[ g_v(u) = f(u) - f(v) - (Tv - w, v - u) \]
is also convex and lower semicontinuous. Hence the set \( \{ u | g_v(u) \leq 0 \} \) is closed and convex. By Lemma 1, however, \( A_w = \bigcap_{v \in D(T)} \{ u | g_v(u) \leq 0 \} \). Hence \( A_w \) is closed and convex for each \( w \).

Suppose now that \( T \) is strictly monotone, and let \( u_1 \) and \( u_2 \) be two elements in \( A_w \). Then
\[ (Tu_1 - w, u_2 - u_1) \geq f(u_1) - f(u_2) \]
\[ (Tu_2 - w, u_1 - u_2) \geq f(u_2) - f(u_1). \]

If we add the two inequalities, we obtain
\[ (Tu_2 - Tu_1 u_2 - u_1) \leq 0, \]

which implies that \( u_1 = u_2 \). q.e.d.

**Lemma 2.** Let \( F \) be a finite-dimensional Banach space, \( T \) a continuous mapping of \( F \) into \( F^* \), \( f \) a lower semicontinuous convex function from \( F \) to \( (-\infty, +\infty] \) with \( f(0) = 0 \). Let \( R > 0 \) be given, \( B_R(F) = \{ u | u \in F, \| u \| \leq R \} \).

Then there exists \( u_0 \) in the ball \( B_R(F) \) such that
\[ (Tu_0 - w, v - u_0) \geq f(u_0) - f(v), \quad v \in B_R(F). \]

**Proof of Lemma 2:** Replacing \( Tu \) by \( Tu - w \), we may assume without loss of generality that \( w = 0 \). If the conclusion of Lemma 2 were false, then for each \( u \) in \( B_R(F) \) there would exist an element \( v \) in \( B_R(F) \) such that
\[ (Tu, v - u) < f(u) - f(v). \quad (8) \]

For a fixed \( v \) in \( B_R(F) \), the set of all \( u \) in \( B_R(F) \) for which the inequality (8) holds is open in \( B_R(F) \) because of the lower semicontinuity of \( f \). (If \( f(v) = +\infty \), we take the set of such \( u \) to be empty.) Hence by the compactness of \( B_R(F) \), we can find a finite set \( \{ v_1, \ldots, v_l \} \) in \( B_R(F) \) such that the corresponding sets
\[ U_j = \{ u | u \in B_R(F), (Tu, v_j - u) < f(u) - f(v_j) \} \]
form a covering of \( B_R(F) \).
Let \( \{ \beta_1, \ldots, \beta_s \} \) be a partition of unity on \( B_R(F) \) corresponding to the covering \( \{ U_1, \ldots, U_s \} \) such that for all \( u \) in \( B_R(F) \), \( 0 \leq \beta_j(u) \leq 1 \), \( \sum_{j=1}^s \beta_j(u) = 1 \). We define
\[
q(u) = \sum_{j=1}^s \beta_j(u)v_j.
\]
Since \( q(u) \) is a convex linear combination of the \( v_j \) with continuous coefficients, \( q \) yields a continuous mapping of \( B_R(F) \) into itself.

For \( u \) in \( B_R(F) \), we have
\[
(Tu, q(u) - u) = \sum_{j=1}^s \beta_j(u)(Tu, v_j - u) < \sum_{j=1}^s \beta_j(u)[f(u) - f(v_j)] = f(u) - \sum_{j=1}^s \beta_j(u)f(v_j),
\]
while by convexity of \( f \),
\[
f(q(u)) \leq \sum_{j=1}^s \beta_j(u)f(v_j).
\]
Hence for all \( u \) in \( B_R(F) \),
\[
(Tu, q(u) - u) < f(u) - f(q(u)).
\]
However, \( q \) has a fixed point \( u_1 \) in \( B_R(F) \) by the Brouwer fixed-point theorem. For this point, we have
\[
0 = (Tu_1, q(u_1) - u_1) < f(u_1) - f(q(u_1)) = 0,
\]
a contradiction proving Lemma 2. q.e.d.

Lemma 3. Let \( T \) be a mapping from a convex domain \( D(T) \) in \( X \) to \( X^* \), \( f \) a convex function from \( X \) to \( (-\infty, +\infty] \) with \( f(0) = 0 \), \( R \) a given positive number. Let \( u_0 \in B_R \cap D(T) \) be a solution of the system of inequalities
\[
(Tu_0 - w, v - u_0) \geq f(u_0) - f(v), \quad v \in D(T) \cap B_R.
\]
Suppose further that for \( u \) in \( D(T) \) with \( \|u\| = R \),
\[
(Tu - w, u) + f(u) > 0.
\]
Then \( \|u_0\| < R \) and \( u_0 \) satisfies the stronger system of inequalities
\[
(Tu_0 - w, v - u_0) \geq f(u_0) - f(v), \quad v \in D(T).
\]
Proof of Lemma 3: We show first that \( \|u_0\| < R \). If we set \( v = 0 \) in the inequality (9), we see that
\[
-(Tu_0 - w, u_0) \geq f(u_0),
\]
i.e., \( (Tu_0 - w, u_0) + f(u_0) \leq 0 \). It follows from the hypothesis that \( \|u\| < R \).

Let \( v \) be any element of \( D(T) \). Then there exists \( t > 0 \) such that the element \( v_t = (1 - t)u_0 + tw \) lies in \( B_R \), while \( v_t \) lies in \( D(T) \) by the convexity of the latter set. Setting \( v_t \) for \( v \) in the inequality (9), we obtain
\[
t(Tu_0 - w, v - u_0) \geq f(u_0) - f(v_t) \geq f(u_0) - \frac{1}{1 - t}f(u_0) + tf(v) = t[f(u_0) - f(v)].
\]
Cancelling \( t > 0 \), we obtain the inequality (10). q.e.d.

Proof of Theorem 1: We may assume without loss of generality that \( w = 0 \). Let \( S \) be the directed set of finite-dimensional subspaces \( F \) of \( X \) ordered by inclusion.
For each $F$ in $S$, let $j_F$ be the inclusion map of $F$ in $X$, $j_F^*$ the projection map of $X^*$ on $F^*$. We define the mapping $T_F$ of $F$ into $F^*$ by

$$T_F = j_F^* T j_F.$$  

Then for each $u$ and $v$ in $F$, $(T_F u, v) = (Tu, v)$ so that $T_F$ is monotone and for $u$ in $F$ with $\|u\| = R$, $(T_F u, u) + f(u) > 0$. Since $T_F$ is monotone and hemicontinuous while $F$ is of finite dimension, $T_F$ is continuous from $F$ to $F^*$.

Hence, applying Lemma 2, there exists, $u_F$ in $F$ with $u_F \in B_R(F)$ such that for all $v$ in $B_R(F)$

$$(T_F u_F, v - u_F) = (Tu, v - u_F) \geq f(u_F) - f(v).$$

Applying Lemma 3, we see that $u_F$ satisfies the stronger system

$$(Tu, v - u) \geq f(u) - f(v), \quad v \in F. \tag{11}$$

For each $F_0$ in $S$, we set $V_{F_0} = \bigcup \{U_F\}$. The sets $V_{F_0}$ have the finite intersection property and are subsets of the weakly compact set $B_R$. Hence there exists an element $u_0$ in $B_R$ which lies in the weak closure of every $v_F$.

Let $v$ be an arbitrary element of $X$, and choose an element $F_0$ of $S$ which contains $v$. For any $u_F$ in $V_{F_0}$, i.e., for any $F$ with $F_0 \subset F$, we have $v \in F$ and hence

$$(Tu, v - u) \geq f(u) - f(v).$$

Since $T$ is monotone,

$$(Tv, v - u) \geq (Tu, v - u) \geq f(u) - f(v).$$

On the other hand, the function $g_r(u)$ of $u$ given for fixed $v$ by

$$g_r(u) = f(u) - f(v) - (Tv, v - u)$$

is lower semicontinuous and convex in $u$. Hence the set $\{u \in B_R : g_r(u) \leq 0\}$ is closed and convex, and hence weakly closed. Since $V_{F_0}$ lies in this set, it follows that $u_0$ must also, and therefore we have

$$(Tv, v - u_0) \geq f(u_0) - f(v), \quad v \in X.$$  

Applying Lemma 1, we see that $u_0$ is a solution of the inequality (2). q.e.d.

**Proof of Theorem 3:** Under the hypotheses of Theorem 3, there exists a function $c(r)$ with $c(r) \to +\infty$ as $r \to +\infty$ such that

$$(Tu, u) + f(u) \geq c(\|u\|) \|u\|.$$  

Hence for a given $w$ in $X^*$,

$$(Tu, u) + f(u) \geq \|w\| \|u\| > 0$$

for $u \geq R(\|w\|)$. Thus by Theorem 1, $A_w \neq \phi$ for each $w$ in $X^*$. q.e.d.

**Theorem 4.** Let $X$ be a reflexive Banach space, $T$ a densely defined mapping from $X$ to $X^*$, $f$ a lower semicontinuous convex function from $X$ to $(-\infty, +\infty]$ with $f(0) = 0$. Suppose that $T = L + T_0$, where:

1. $T_0$ is a hemicontinuous monotone mapping from all of $X$ to $X^*$ which carries bounded sets into bounded sets.
2. $L$ is a closed densely defined monotone linear operator from $X$ to $X^*$ with the
property that if for a given element \( u_0 \) of \( X \) with \( f(u_0) \) finite and each constant \( N > 0 \), there exists a constant \( M_N > 0 \) such that

\[
(u_0, L^*v) \geq - M_N(\|v\| + 1)
\]

for all \( v \) in \( D(L) \cap D(L^*) \cap \{ v | f(v) \leq N \} \), then \( u_0 \) lies in \( D(L) \).

(3) For a given \( w \) in \( X^* \), there exists \( R > 0 \) such that

\[
(Tu - w, u) + f(u) > 0
\]

for all \( u \) in \( D(T) = D(L) \) with \( \|u\| < R \).

Then there exists \( u_0 \) in \( D(T) \) with \( \|u_0\| < R \) such that

\[
(Tu_0 - w, v - u_0) \geq f(u_0) - f(v), \quad v \in D(T).
\]

(12)

Proof of Theorem 4: It suffices to assume \( w = 0 \). Let \( S \) be the directed set of finite-dimensional subspaces of \( D(T) = D(L) \) ordered by inclusion. As in the proof of Theorem 1, there exists \( u_F \) in \( F \) with \( u_F e \in BR(F) \) such that

\[
(Tu_F, v - u_F) > f(u_F) - f(v), \quad v \in F.
\]

If \( V_F = \bigcup_{F_0 \subset F} \{ u_F \} \), there exists \( u_0 \) in \( BR \) which lies in the weak closure of \( V_F \) for each \( F_0 \) in \( S \).

Let \( v \) be any element of the set \( D(L) \cap D(L^*) \cap \{ v | f(v) \leq N \} \) for a given \( N > 0 \), and choose an \( F_0 \) in \( S \) which contains \( v \). For any \( u_F \) in \( V_F \), we have

\[
(Lu_F, v - u_F) > f(u_F) - f(v) - (Tu_F, v - u_F).
\]

Since \( f(u_F) \) is uniformly bounded for all \( u_F \) in \( V_F \), it follows that \( f(u_0) \) is finite. Moreover, we have

\[
(Lu_F, v) \geq (Lu_F, u_F) - \| Tu_F \| \{ \| u_F \| + \| v \| \} + f(u_F) - N.
\]

Since \( f \) is convex and lower semicontinuous, it is bounded from below on \( BR \). Since the elements \( u_F \) lie in \( BR, \| Tu_F \| \) is uniformly bounded. Hence,

\[
(u_F, L^*v) = (Lu_F, v) - c_1(R) - c_2(R) \times \{ \|v\| + R \} - N = - c(R) \{ \|v\| + R \} - N.
\]

Since \( u_0 \) lies in the weak closure of \( V_F \), it follows that

\[
(u_0, L^*v) \geq - c(R) \{ \|v\| + R \} - N,
\]

and hence by hypothesis (2) of Theorem 4, \( u_0 \) lies in \( D(T) \).

If now we let \( v \) be any element of \( D(T) \) and choose \( F_0 \) to be an element of \( S \) which contains \( v \), then for any \( u_F \) in \( V_{F_0} \):

\[
(Tu_F, v - u_F) \geq f(u_F) - f(v).
\]

Since \( T \) is monotone,

\[
(Tv, v - u_F) \geq (Tu_F, v - u_F) \geq f(u_F) - f(v).
\]

The function \( g_s(u) \) of \( u \) given by

\[
g_s(u) = f(u) - f(v) - (Tv, v - u)
\]

is convex and lower semicontinuous and hence weakly lower semicontinuous.
Since \( g_s(u) \leq 0 \) for \( u \) in \( V_{F_s} \) by the preceding inequality, it follows that \( g_s(u_0) \leq 0 \), i.e.,

\[
(T_v v - u_0) \geq f(u_0) - f(v), \quad v \in D(T).
\]

Applying Lemma 1, we see that since \( u_0 \in D(T) \):

\[
(Tu_0 v - u_0) \geq f(u_0) - f(v), \quad v \in D(T).
\]

(12)

q.e.d.

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3 This fact was first observed by R. I. Kacurovski in Uspekhi Mat. Nauk, 15, 213–215 (1960), where the explicit definition of monotone operator was first given.


