ON VISUAL HULLS OF SETS*

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1. The problems treated in this note concern the possibilities of "reconstructing" a subset $S$ of a real Euclidean $n$-space $E_n$ ($n \geq 2$) or, more generally, a real Hilbert space $H$, from the knowledge of the orthogonal projections of $S$ onto each member of a preseleced family $\mathfrak{M}$ of proper subspaces of $E_n$ (or of $H$). Obviously, in general, this is not possible: The surface of a sphere in $E_3$ has the same projections on each two-dimensional subspace as the solid sphere. We introduce, however, the notion of the "largest" set $\Phi(S)$ which has the same projections as $S$ (with respect to the given family $\mathfrak{M}$). We call this set $\Phi(S)$ the visual hull of $S$, and identify by the equation $\Phi(S) = S$ those subsets $S$ of $H$ which can be "reconstructed" from a knowledge of their orthogonal projections onto the members of $\mathfrak{M}$. Although there are many interesting choices for $\mathfrak{M}$, we mention three in particular:

1. all one-dimensional subspaces of $H$,
2. all two-dimensional subspaces of $H$ (when dim $H \geq 3$),
3. all hyperplanes (= closed maximal proper subspaces of $H$).

The visual hull $\Phi(S)$ is related to the notion of the convex hull $\mathcal{C}(S)$ in the following way. If $\mathcal{M}$ is the family of all one-dimensional subspaces [case (1) above], then for every connected set $S$ it turns out that $\mathcal{C}(S) \subseteq \Phi(S) \subseteq \mathcal{C}(S)$. However, if $S$ is not connected, then $\Phi(S)$ is only a subset of $\mathcal{C}(S)$. On the other hand, for other choices of $\mathfrak{M}$ [such as in cases (2) and (3) above] $\Phi(S)$ need not be convex even when $S$ is connected. For example, if $S$ is an anchor-ring (solid torus) in $E_3$ and $\mathfrak{M}$ is as in case (2) above, then $\Phi(S)$ is larger than $S$ but still homeomorphic to $S$.

Case (2) above is obviously suited to problems of "photographic recognition" or "graphic representation."

In this note we present a precise formulation of these problems and some first results of a general nature.

2. Let $H$ denote a real Hilbert space of any finite or transfinite dimension with inner product $(x,y)$ and norm $\|x\| = (x,x)^{1/2}$. Given a family $\mathfrak{P}$ of nonzero orthogonal projections (i.e., linear, symmetric, idempotent mappings $P:H \to H$), we say that two subsets $A$ and $B$ of $H$ are $\mathfrak{P}$-equivalent, written $A \equiv B$, if $P(A) = P(B)$ for all $P \in \mathfrak{P}$. This is obviously an equivalence relation for the subsets of $H$. For each subset $S$ of $H$ we define $\Phi(S)$ to be the set-union of all subsets $B$ of $H$ satisfying $B \subseteq S$. Obviously $S \subseteq \Phi(S)$.

**Theorem 1.** $S \subseteq \Phi(S)$, $\Phi(\Phi(S)) = \Phi(S)$, and $\Phi_1 \subseteq \Phi_2 \Rightarrow \Phi_2(S) \subseteq \Phi_1(S)$.

**Proof:** First, $S \subseteq \Phi(S)$ implies $P(S) \subseteq P(\Phi(S))$ for all $P \in \mathfrak{P}$. On the other hand, if $x \in P(\Phi(S))$, then $x = P(a)$ for some $a \in A$, where $A \subseteq S$. But then $P(a) = P(s)$ for some $s \in S$, so that $x = P(s)$, or $x \in P(S)$. The second equation follows from the first and the definition of $\Phi(S)$, since $\equiv$ is an equivalence relation. The last assertion of the theorem is just as obvious.

Thus for each subset $S$ of $H$, $\Phi(S)$ is the largest subset of $H$ which is $\mathfrak{P}$-equivalent to $S$. We are particularly interested in those subsets $S$ of $H$ satisfying $S = \Phi(S)$, for it is these subsets of $H$ which could be considered as "completely determined" or
"reconstructible" from a knowledge of their \(\Phi\)-projections. By "an affine subspace of \(H\" we shall mean a translated subspace of the form \(x + M\). If \(P\) is a projection, \(\Phi(P)\) denotes its range. If \(A \subset H, A^\perp = \{x \in H: \langle x, a \rangle = 0 \text{ for all } a \in A\\}.

**Theorem 2.** For each subset \(S\) of \(H, x \in \Phi(S)\) iff for every \(P \in \Phi\) the affine subspace \(x + \Phi(P)^\perp\) intersects \(S\) in at least one point.

**Proof:** \(x \in \Phi(S)\) iff \(x \in A\) for some \(A \supseteq \Phi(S)\). The latter condition is equivalent to the statement that for every \(P \in \Phi\), \(P(x) = P(s)\) for some \(s \in S\) [for then \(A = \{x\} \cup S\) is \(\Phi\)-equivalent to \(S\)] Now if \(P(s) = P(x)\), then \(s = x + (I - P)(s - x)\) which belongs to \(x + \Phi(P)^\perp\). Conversely, if \(s \in x + \Phi(P)^\perp\) for some \(s \in S\), then \(s = x + (I - P)(s - x)\) so that \(P(s) = P(x)\).

**Corollary 1.** \(S = \Phi(S)\) iff for every \(x \notin S\) there exists a \(P \in \Phi\) such that the affine subspace \(x + \Phi(P)^\perp\) does not intersect \(S\).

A family \(\Phi\) will be called full if every one-dimensional subspace of \(H\) is contained in the range of at least one member of \(\Phi\). This property may be satisfied by \(\Phi\) in many different ways. In particular, it is satisfied if \(\Phi\) is the family of projections corresponding to any one of the three cases mentioned in the introduction.

**Corollary 2.** If \(\Phi\) is a full family of projections, then for every subset \(S\) of \(H, \Phi(S)\) is contained in the closed convex hull \(\mathcal{E}(S)\) of \(S\).

**Proof:** If \(x \notin \mathcal{E}(S)\), then by the Hahn-Banach theorem there exists a closed hyperplane \(M\) of \(H\) so that \(x + M\) does not intersect \(\mathcal{E}(S)\). A fortiori, \(x + M\) does not intersect \(S\). By hypothesis there exists a \(P \in \Phi\) such that \(M^\perp \subset \Phi(P)\). For this \(P \in \Phi\), \(x + \Phi(P)^\perp\) does not intersect \(S\). Consequently, by Theorem 2, \(x \notin \Phi(S)\).

**Corollary 3.** If \(\Phi\) is a full family of projections, then every closed convex subset \(S\) of \(H\) satisfies \(S = \Phi(S)\).

**Corollary 4.** If \(\Phi\) is exactly the family of orthogonal projections onto all one-dimensional subspaces of \(H\) and if \(S\) is a connected subset of \(H\), then \(\mathcal{E}(S) \subset \Phi(S) \subset \mathcal{E}(S)\); in particular \(\Phi(S)\) is convex.

**Proof:** \(\Phi(S) \subset \mathcal{E}(S)\) by Corollary 2. If \(x \notin \Phi(S)\), then there exists (by Theorem 2) a \(P \in \Phi\) such that \(x + \Phi(P)^\perp\) does not intersect \(S\). But since \(\Phi(P)\) is one-dimensional, \(x + \Phi(P)^\perp\) is a closed affine hyperplane which separates \(H\) into two open half spaces \(H_+\) and \(H_-\). Since \(S \subset H_+ \cup H_-\) and \(S\) is connected, it follows that \(S\) is completely contained in one of the half spaces or the other, say \(H_+\). But these half spaces are convex so that \(\mathcal{E}(S) \subset H_+\). It follows now that \(x \notin \mathcal{E}(S)\). Therefore we have shown that \(\mathcal{E}(S) \subset \Phi(S)\), and this completes the proof.

**Theorem 3.** If \(S = A_1 \cup \cdots \cup A_m\) for some integer \(m \geq 1\) where each \(A_k\) is convex and either open or closed (but not necessarily disjoint from the other \(A_k\)'s) and if each \(m\)-dimensional subspace of \(H\) is contained in the range of at least one member of \(\Phi\) (so that, in particular, we must assume \(\dim H > m\)), then \(\Phi(S) = S\).

**Proof:** Suppose \(x \notin S\). Then for each \(k = 1, \ldots, m\) it follows from the Hahn-Banach theorem that there exists a closed hyperplane \(M_k\) such that \(x + M_k\) does not intersect \(A_k\). Let \(L = \bigcap_{k=1}^m M_k\). Then \(L\) is a closed subspace of \(H, L^\perp\) has dimension \(\leq m\), and \(x + L\) does not intersect \(S\). By the hypothesis of the theorem there exists a projection \(P \in \Phi\) such that \(L^\perp \subset \Phi(P)\) and therefore \(x + \Phi(P)^\perp \subset x + L\) so that \(x + \Phi(P)^\perp\) does not intersect \(S\). But then by Corollary 1 \(\Phi(S) = S\), as was to be shown.
If $S \subseteq E_2$, $\Phi$ is the family of projections onto all one-dimensional subspaces of $E_2$, and $S$ is the union of two disjoint convex sets, then $\Phi(S)$ is also the union of two convex sets which, however, may be larger than the two given sets. An analogous theorem is presumably valid in arbitrary $E_n$ for $n$ disjoint convex sets. We remark that given a set $S$, there exist besides $\Phi(S)$ "smallest" (i.e., minimal) sets which have the same projections as the given set, but these are not unique and their existence depends on Zorn's lemma.

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