THEORY OF THE PHYSICAL CHARACTERISTICS OF TENSILE FRACTURE UNDER PRESSURE*

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(1) Bridgman's Observational Results.—It was observed by Bridgman that there is a linear relationship between the hydrostatic pressure \( p \) and the natural strain at fracture, i.e., the ductility, in tensile tests on steel cylinders, and that this relationship persists regardless of the type of fracture which occurs. A theoretical explanation of the foregoing result has been given in an earlier paper in these PROCEEDINGS.\(^2\)

When fracture takes place at atmospheric pressure, it is almost always of the cup-cone type and the flat bottom or tensile part of the break is accurately situated along the minimum section of the neck. The line of separation of the flat bottom and the sides of the crater is circular and is usually quite definitely defined; hence, one can determine the area of the flat bottom as well as the area of the minimum section of the neck of the fractured cylinder. The ratio of the area of the flat bottom to the area of the minimum section after fracture appears to be one of the characteristics of the steel. Bridgman found that the effect of pressure is to diminish the ratio of these areas. His results for the steel designated as C14 are represented graphically in Figure 1. Analogous results were obtained for steels in the 19-1, . . ., 19-4 series; the data for these steels is given in the table and is used for the construction of the graphs in Figure 2. It appears from

these graphs that the relationship between the ratio of the areas and the pressure is linear to a close approximation.

At higher pressures the flat bottom of the crater, or tensile part of the fracture, will disappear, and the break will be entirely shearing in character. The graphs in Figures 1 and 2 show the minimum pressures capable of producing such shearing fracture for the steel C14 and steels in the 19-1, . . ., 19-4 series.

It is the purpose of the following to give a simple theoretical explanation of the
above results which express the main features of the tensile fracture of steel cylinders under pressure as observed by Bridgman.

(2) The Stress Field Preceding Fracture.—The following assumption will be made. Immediately preceding the occurrence of fracture there will be an abrupt change in the stress resulting in a new stress field, hereafter referred to as the fracture field, which will play the essential role in the determination of the physical characteristics of the fracture under consideration. The origin of the fracture field may be attributed to the changes in stress due to the over-all release and redistribution of internal energy combined with microscopic rupture and self-healing which mark the first stage of the fracture process (ref. 1, pp. 351 and 352). A further release of internal energy may, of course, occur during the final stage of fracture in which the body separates into distinct parts as the result of crack propagation.

It will suffice for our purpose to consider the fracture field over the minimum section of the neck of the cylinder. If we denote by \( \sigma_{11} \) the stress components of the fracture field in the cylindrical coordinate system to which the cylinder is referred, it will be assumed (1) that the quantities \( \sigma_{11} \) and \( \sigma_{22}/r^2 \) are equal, and (2) that the difference \( \sigma_{33} - \sigma_{11} \) is constant over the minimum section. These are precisely the conditions imposed by Bridgman (ref. 1, p. 13) in his derivation of the plastic stress field over the minimum section of the neck; however, the second of the above assumptions will now be interpreted in a more restrictive sense than was done by Bridgman, i.e., it will be supposed that the difference \( \sigma_{33} - \sigma_{11} \) is an absolute constant or modulus of the material, and hence this difference will be independent of the particular minimum section which exists at the initiation of the fracture field. Denoting the above modulus by \( \lambda \), we can therefore write

\[
\sigma_{11} = \sigma_{22}/r^2; \quad \sigma_{33} - \sigma_{11} = \lambda. \tag{1}
\]

The remaining components \( \sigma_{12}, \sigma_{33}, \) and \( \sigma_{23} \) vanish from symmetry over the minimum section of the cylinder.

Now consider the equilibrium equations which reduce to

\[
\frac{\partial \sigma_{11}}{\partial r} + \frac{\partial \sigma_{12}}{\partial z} + \frac{\sigma_{11} - \sigma_{22}/r^2}{r} = 0,
\]

\[
\frac{\partial \sigma_{12}}{\partial r} + \frac{\partial \sigma_{33}}{\partial z} + \frac{\sigma_{12}}{r} = 0,
\]
in view of the symmetry conditions of the problem. On the minimum section of the neck these equations become

\[ \frac{\partial \sigma_{11}}{\partial r} + \frac{\partial \sigma_{13}}{\partial z} = 0; \quad \frac{\partial \sigma_{33}}{\partial z} = 0. \]  

(2)

The second equation (2) provides no useful information concerning the fracture field over the minimum section, but significant information can be obtained from the first of these equations provided we can find a suitable expression for the derivative \( \frac{\partial \sigma_{13}}{\partial z} \) on the minimum section of the neck. In this connection we observe that the quantity \( \sigma_{13} \) gives the radial component of the shearing stress on a surface element normal to the central or \( z \) axis of the cylinder. Using this fact it is easily seen from symmetry and the continuity of the component \( \sigma_{13} \) that \( \sigma_{13} \) must vanish along the \( z \) axis. Hence the derivative \( \frac{\partial \sigma_{13}}{\partial z} \) must vanish on the \( z \) axis and, in particular, this derivative must vanish at the point \( r = 0 \) on the minimum section of the neck of the cylinder.

Let us now make the following assumption. The derivative \( \frac{\partial \sigma_{13}}{\partial z} \) is a linear homogenous function of the radius \( r \) over the minimum section of the neck. This assumption gives the simplest expression for the derivative \( \frac{\partial \sigma_{13}}{\partial z} \) which satisfies the condition at the end of the preceding paragraph and which also offers the possibility of being valid to within a satisfactory degree of approximation. More specifically, it will be assumed that

\[ \frac{\partial \sigma_{13}}{\partial z} = \frac{2mr}{a^2}, \]  

(3)

over the minimum section of the neck where \( a \) is the radius of the minimum section and \( m \) is a constant or modulus of the material. Substituting the expression (3) for \( \frac{\partial \sigma_{13}}{\partial z} \) into the first equation (2) and integrating subject to the condition \( \sigma_{11} = -p \) for \( r = a \), we find that

\[ \sigma_{11} = \frac{\sigma_{22}}{r^2} = -p + m \left( 1 - \frac{r^2}{a^2} \right), \]  

(4)

\[ \sigma_{33} = -p + m \left( 1 - \frac{r^2}{a^2} \right) + \lambda, \]  

(5)

when account is taken of the equations (1). We have now arrived at the following result. The equations (4) and (5) give the principal stresses in the fracture field over the minimum section of the neck.

It will be shown in the next section that the above fracture stresses lead to a linear relation between the ratio of the areas, discussed in section (1), and the pressure; the precise graphs in Figures 1 and 2 can be obtained from this relation. Hence it would seem reasonable to suppose that the equations (4) and (5) give the principal stresses in the fracture field over the minimum section of the neck to an approximation which is roughly equal to the approximation involved in the linear relationship between the ratio of the areas and the hydrostatic pressure.
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Tensile Fracture under Pressure.—It is assumed that there will be a tendency for fracture to occur over a surface element if the inequality

\[ S + qN \geq Q \]  

is satisfied where \( S(\geq 0) \) and \( N \) are the shearing and normal stresses on the element; the quantities \( q \) and \( Q \) in this condition are the fracture moduli of the solid. \(^2\) A surface \( \Sigma \) over which the inequality (6) is satisfied will be a possible surface of fracture if the surface extends to a free boundary of the solid so that the conditions for the realization of fracture will be favorable. The surface \( \Sigma \) may be a regular surface, or it may consist of the union of two or more regular surfaces \( \Sigma' \). In the latter case the part of the fracture which occurs over any subsurface \( \Sigma' \) of the complete fracture surface \( \Sigma \) will be said to be of tensile type or to be a tensile fracture provided \( S = 0 \) and \( qN > Q \) over \( \Sigma' \); otherwise, i.e., if (6) holds with \( S > 0 \), the fracture will be called a shearing fracture over \( \Sigma' \).

The complete fracture surface \( \Sigma \) cannot be determined theoretically for the problem under consideration until we have some knowledge of the fracture field off the minimum section of the neck. Since this knowledge is lacking at present, we shall make the following assumption. A cup-cone type of fracture will occur with the flat bottom lying on the minimum section of the neck for some range of values of the hydrostatic pressure. On the basis of this assumption, which is in strict accord with Bridgman’s observations, we can proceed to the determination of the relation between the ratio of the areas and the pressure.

It is immediately evident that \( S = 0 \) and \( N = \sigma_{33} \) over the minimum section of the neck. Hence, if fracture occurs over the minimum section, it must be of the tensile type, and the condition for such fracture will be given by

\[ q\left[ -p + m \left( 1 - \frac{r^2}{a^2} \right) + \lambda \right] \geq Q, \]  

when use is made of the equation (5) for the stress component \( \sigma_{33} \). Now suppose that the strict equality (7) holds for \( r = \bar{r} \), i.e., that

\[ q\left[ -p + m \left( 1 - \frac{\bar{r}^2}{a^2} \right) + \lambda \right] = Q. \]  

But the bracketed expression in the left member of (7) is a decreasing function of \( r \) and hence the equation (8) implies

\[ q\left[ -p + m \left( 1 - \frac{r^2}{a^2} \right) + \lambda \right] > Q, \quad \text{for } 0 \leq r < \bar{r}, \]

\[ q\left[ -p + m \left( 1 - \frac{r^2}{a^2} \right) + \lambda \right] < Q, \quad \text{for } \bar{r} < r \leq a. \]

In other words, tensile fracture will occur over the minimum section within a circle whose radius \( \bar{r} \) is given by (8), and for \( r > \bar{r} \) the fracture will be of the shearing type, i.e., the lip of the cup-cone fracture will correspond to values of \( r > \bar{r} \) in accordance with the above italicized assumption.

Solving the equation (8) for the ratio \( \frac{\bar{r}^2}{a^2} \), we have
\[ \frac{r^2}{a^2} = -\frac{p}{m} + (1 - \alpha), \]  

(9)

where \( \alpha \) is a modulus of the solid such that

\[ \alpha = \frac{1}{m} \left( \frac{Q}{q} - \lambda \right). \]

Since \( \frac{r^2}{a^2} \) is the ratio of the areas in question, the following result can be stated.  *The ratio of the areas is given in terms of the hydrostatic pressure \( p \) by the equation (9).*  By choosing appropriate values of the moduli \( m \) and \( \alpha \) the linear graphs shown in Figures 1 and 2 can be obtained.  It is also seen from the equation (9) that for \( p \geq (1 - \alpha)m \) the cup-cone type of fracture will not be possible, and one is therefore led to conclude that for these values of the pressure the fracture must be entirely of the shearing type as observed by Bridgman.

Finally, I would like to call attention to the fact that the flat bottom of the fracture surface of a cylinder which has experienced fracture of the cup-cone type is always more or less coarse-grained in texture.  This may be explained as a deviation from strictly tensile fracture due to the occurrence of flaws in the material.  Such effects are natural and in no way refute the underlying validity of the theory.

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2 Thomas, T. Y., “Effect of pressure on the ductility of solids,” these *PROCEEDINGS*, 58, 1274 (1967).