STRESS-STRAIN RELATIONS FOR CRYSTALS CONTAINING PLASTIC DEFORMATION*

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Communicated May 29, 1968

Let \( P \) be one of the possible slip planes in a crystal, and let \( \lambda \) be one of the possible slip directions in the plane \( P \). Also let \( S \) be the magnitude of the shearing stress on the plane \( P \) resolved in the direction \( \lambda \), and \( S_0 \) the critical value of \( S \), i.e., the least value of \( S \) sufficient to permit slip or plastic deformation under the conditions specified in the following italicized statement; if such slip is realized, it will be considered to occur over the plane \( P \) and in the direction \( \lambda \), as is commonly assumed. The ratio \( S/S_0 \) will, of course, depend on the plane \( P \) as well as the selection of the direction \( \lambda \), and the values of the various ratios \( S/S_0 \) will not be constant throughout the crystal except in the case of a uniform stress field. Denoting by \( \max S/S_0 \) the greatest value of the possible ratios \( S/S_0 \) at points of the crystal, we shall now make the following assumption. Slip or plastic deformation will occur in a region \( R \) of the crystal if

\[
\max S/S_0 \geq 1
\]

*everywhere in the region \( R \).* If \( R \) is an interior region of the crystal, slip will be restricted by the surrounding medium which must remain in the elastic state. This restriction will be removed if \( R \) contains a free boundary of the crystal; in such regions \( R \), comparatively large plastic deformation, as well as the possibility of sustained plastic flow, may occur. In writing the slip condition (1), it is assumed that the normal stresses on the slip planes are negligibly small. Otherwise condition (1) is closely analogous to the condition for slip which I have previously used to account for Bridgman's observational results on the flow and fracture characteristics of solids under hydrostatic pressures.¹

The following assumption will be made. In any crystallographic region \( R \) containing plastic deformation, the stress-strain relations can effectively be represented by an extension of the corresponding relations of the classical theory of elasticity, namely,

\[
\sigma_{ij} = \xi \epsilon_{ki} \delta_{ij} + 2\eta \epsilon_{ij}
\]

(2)

in which the coefficients \( \xi \) and \( \eta \) are suitable scalar invariants of the stress field. The quantities \( \sigma_{ij} \) and \( \epsilon_{ij} \) are the components of the stress and strain tensors, respectively, in a system of rectangular coordinates to which the relations (2) are obviously referred; the components of the latter tensor are given by

\[
\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),
\]

where the \( u_i \) are the components of the deformation and the comma denotes partial differentiation with respect to the coordinates. Due regard must, of course, be taken of the constraints imposed by the crystalline structure, e.g., the condition that slip may occur only over certain planes in the crystal, in the application of the stress-strain relations (2).
It is natural to suppose that the scalars $\xi$ and $\zeta$ are functions of the quantity $\max S/S_0$ because of the basic relationship between this quantity and plastic deformation in the crystal. Hence, we shall assume that

$$\xi = \xi(S/S_0); \quad \zeta = \zeta(S/S_0),$$

(3)

where we have replaced $\max S/S_0$ by $S/S_0$ for simplicity; also, the symbols $S$ and $S_0$ will be used in the following discussion to refer to those quantities $S$ and $S_0$ for which the ratio $S/S_0$ has its maximum value at points of the region $R$ containing plastic deformation. In addition to the ratio $S/S_0$ which appears explicitly in the equations (3), it is to be understood that the right members of these equations may involve implicitly one or more constant moduli of the crystal. The coefficients $\xi$ and $\zeta$ in the stress-strain relations (2) are therefore material functions which are given by equations of the form (3), and the relations (2) are applicable in any region $R$ admitting slip or plastic deformation as determined by the slip condition (1). An article to appear in the Journal of Mathematics and Mechanics will show that two very simple functions $\xi$ and $\zeta$ will suffice to account for the velocities of the edge and screw dislocations which were observed by Johnston and Gilman2 in their experiments on lithium fluoride crystals. In the remainder of this note, we shall show that the form of the relations (2) can be modified so as to express the components $e_{ij}$ in terms of the components $\sigma_{ij}$ by a system of nonlinear stress-strain relations. Also, the components $\sigma_{ij}$ can be expressed in terms of the components $e_{ij}$ in a corresponding manner.

Putting $i = j$ in the relations (2) and summing on these repeated indices, we obtain

$$\sigma_{kk} = (3\xi + 2\zeta)\varepsilon_{kk}.$$  

(4)

Elimination of the quantity $\varepsilon_{kk}$ from (2) by means of (4) leads to the equations

$$e_{ij} = \frac{-\xi\sigma_{kk}\delta_{ij}}{2\zeta(3\xi + 2\zeta)} + \frac{\sigma_{ij}}{2\zeta},$$

(5)

in which the scalars $\xi$ and $\zeta$ are functions of the ratio $S/S_0$ in accordance with the equations (3). But $S_0$ can be regarded as one of the moduli of the crystal, and the magnitude $S$ of the resolved shearing stress is given by

$$S = \sigma_{i}n^i\lambda^j,$$

(6)

provided that the direction of the unit vector $n$ normal to the slip plane $P$ and the direction of the unit vector $\lambda$ (which has already been defined) are such that the right member of this equation is positive; this condition can be met since the direction of each of these vectors can be selected arbitrarily. It follows therefore from (3) and (6) that the quantities $\xi$ and $\zeta$ are functions of the stress components $\sigma_{ij}$, and hence the strain components $e_{ij}$ are expressed in terms of the components $\sigma_{ij}$ of the stress tensor by the nonlinear relations (5).

If we multiply both members of (2) by the quantity $n^i\lambda^j$ and sum on the repeated indices $i$ and $j$, we find that

$$S = 2\zeta(S/S_0)E; \quad E = e_{ij}n^i\lambda^j,$$

(7)
when use is made of (6) and of the fact that the unit vectors \( n \) and \( \lambda \) are perpendicular. The quantity \( E \) in the first equation (7) is defined by the second of these equations and, as so defined, \( E \) is the magnitude of the resolved shearing strain. In particular, it follows from (7) that

\[
S_0 = 2\xi(S_0/S_0)E_0 = 2\xi(1)E_0,
\]

(8)

where \( \xi(1) \) is a constant modulus of the crystal; also \( E_0 \) is the critical shearing strain and is thus a modulus of the crystal analogous to the critical shearing stress \( S_0 \). Hence,

\[
\frac{S}{S_0} = \frac{\xi(S/S_0)}{\xi(1)} \frac{E}{E_0}, \quad \frac{S}{S_0} = F \left( \frac{E}{E_0} \right)
\]

(9)

The first of the equations (9) is obtained by dividing corresponding members of the first equation (7) and equation (8); the second equation (9) is obtained as a solution of the first of these equations. It is easily seen from the derivation of the first equation (9) that the value of the above ratio \( E/E_0 \) is the greatest of the possible values of this ratio at points of the crystal. When we denote this maximum ratio by \( E/E_0 \) corresponding to the use of the symbol \( S/S_0 \) in the above discussion, it follows from (3) and the second equation (9) that the stress-strain relations (2) can be written

\[
\sigma_{ij} = A(E/E_0)e_{ik} \delta_{kj} + 2B(E/E_0)e_{ij},
\]

(10)

where the coefficients \( A \) and \( B \) are functions of the ratio \( E/E_0 \) as indicated; these coefficients will also depend implicitly on the above modulus \( \xi(1) \) as well as on other constant moduli which enter implicitly in the right members of the equations (3). This completes the proof of the above statement that the stress components \( \sigma_{ij} \) can be expressed in terms of the strain components \( e_{ij} \) by a system of nonlinear relations.

Finally, I should like to mention that the stress-strain relations (2) and hence the equivalent relations (5) and (10) are invariant under the group of orthogonal transformations relating the coordinates of arbitrarily moving rectangular coordinate systems. Attention is called to this fact since any system of constitutive equations, e.g., the stress-strain relations (2), should satisfy this theoretical requirement as is generally recognized.

* Supported by the U.S. Army Research Office (Durham).
