ORDERS OF CONVERGENCE OF THE RAYLEIGH-RITZ AND WEINSTEIN-BAZLEY METHODS*

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In this note, a “best possible” order-of-convergence theorem is proved for eigenfunctions computed by the approximate method of Rayleigh-Ritz (ref. 2, pp. 175-176). This result parallels the corresponding result for eigenvalues established by Birkhoff, de Boor, Swartz, and Wendroff.¹ A similar order-of-convergence theorem will be proved for the approximate eigenfunctions and eigenvalues computed by the Weinstein-Bazley method (ref. 3, chap. 12). This latter method yields lower bounds to the eigenvalues, while the Rayleigh-Ritz approximate eigenvalues are upper bounds. These theorems concern perturbations of self-adjoint operators of a kind arising in many concrete Sturm-Liouville systems, and characterized abstractly by the following definition, in which $D(B)$ denotes the range of the operator $B$.

Definition: Self-adjoint operators $L_0$ and $L = L_0 + B$ on a Hilbert space $\mathcal{H}$ will be said to have compatibly compact inverses when:

(a) $L_0$ and $L$ are positive definite and have compact inverses,

(b) $B = L - L_0$ is self-adjoint and $D(B) \supset D(L_0)$.

Note that $L$ and $L_0$ have orthonormal bases of eigenvectors, which we denote by $\{u_k\}$ and $\{v_k\}$, respectively, with eigenvalues which can be ordered as

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots, \quad 0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots,$$

We let $u_k$, $1 \leq k \leq n$, be the $k$th approximate eigenvector obtained by the Rayleigh-Ritz method, i.e. as the (projective) “point” where the “Rayleigh quotient” $(Lu,u)/(u,u)$ assumes its $k$th critical value on the subspace

$$S_n = \text{Span}\{v_j\}_{j=1}^n.$$

Theorem 1. Let $L_0$ and $L_0 + B$ be compatibly compact, and let $Q_n$ be the projection of $\mathcal{H}$ onto $S_n$ with

$$\|B(I - Q_n)u_k\| = O(n^{-r'}), \quad \|(I - Q_n)u_k\| = O(n^{-r^*}),$$

as $n \to \infty$. If also

$$\lambda_{k-1} < \lambda_k < \lambda_{k+r+1},$$

then

$$\text{dist}\{u_k, \text{Span}\{u_j\}_{j=k-r}^{k+r}{_1}\} = O(n^{-r}), \quad r = \min(r', r^*).$$

Corollary. If $\lambda_k$ is simple, then

$$\|u_k - u^n_k\| = O(n^{-r}).$$

Observe that if $B$ is bounded, then $r' = r^* = r$, and (5') states that the Rayleigh-Ritz approximation $u^n_k$ has the same order of accuracy as the best approximation $Q_n u_k$ to $u_k$ in $S_n$.
THEOREM 2. In addition to the hypotheses (a) and (b), assume (c) that $B$ is bounded with $(Bu, u) \geq \lambda \|u\|^2$ for all $u \in \mathcal{C}$. Then $B^{-1}$ exists; hence so does the orthogonal projection $P_\mathcal{C}$ on $\text{Span}\{B^{-1}v_j\}_{j=1}^\ell$ in the Hilbert space $\mathcal{C}'$ associated with the inner product $[u, v] = (Bu, v)$.

Let (3) hold, and let

$$\| (I - Q_\mathcal{C}) Bu_k \| = O(n^{-p}), \quad (L[I - Q_\mathcal{C}] u_k, [I - Q_\mathcal{C}] u_k) = O(n^{-i}),$$

as $n \to \infty$. Then, if $v_k^\mathcal{C}$ and $\mu_k^\mathcal{C}$ are the $k$th orthonormal eigenvector and eigenvalue of the $n$th intermediate Weinstein-Bazley operator (ref. 3, chap. 12)

$$L_\mathcal{C} = L_0 + BP_\mathcal{C},$$

when $\lambda_k$ is simple, we have

$$\| u_k - v_k^\mathcal{C} \| = O(n^{-r}), \quad \lambda_k - \mu_k^\mathcal{C} = O(n^{-2r}), \quad 1 \leq k \leq n,$$

where

$$r' = \min (r, p), \quad r = \min [r, p, t/2].$$

Alternatively, if $\lambda_k$ is a multiple eigenvalue as in (4), then

$$\text{dist} \{ u_k, \text{Span}\{v_k^\mathcal{C}\}_{k=1}^\ell \} = O(n^{-r}), \quad \lambda_k - \mu_k^\mathcal{C} = O(n^{-\ell}).$$

Observe that if $p = r$, then the eigenvector estimate (8) is best possible.

Applications: Typical applications of Theorems 1 and 2 for various special cases are as follows; a detailed discussion will be published elsewhere.

For boundary value problems which have smooth periodic eigenfunctions, such as

$$Lu = -u'' + q(x)u = \lambda u, \quad q(x + \pi) = q(x), \quad q \in C^\infty(-\infty, \infty),$$

subject to periodic boundary conditions, the above (with $L_0 u = -u''$, $Bu = qu$, $q > 0$) implies that both the Rayleigh-Ritz and Weinstein-Bazley approximations have an infinite order of convergence. For regular Sturm-Liouville problems in normal form,

$$Lu = -u'' + q(x)u, \quad u(0) = u(1) = 0,$$

Theorem 1 implies that the Rayleigh-Ritz approximate eigenvalues and eigenfunctions have an error of $O(n^{-7})$ and $O(n^{-7/2})$, respectively; Theorem 2 implies that the Weinstein-Bazley approximations have errors of $O(n^{-8})$ and $O(n^{-5/2})$.

Proof of Theorem 1: The basic idea of the proof is to show that $Q_\mathcal{C} u_k$ is "close" to the $k$th Rayleigh-Ritz approximate eigenvector. This will be accomplished by using the following theorem due to Wilkinson (ref. 4, pp. 172–173).

Theorem W (Wilkinson). Let $A$ be an $n \times n$ real symmetric matrix with eigenvalues $\Lambda_1 \leq \ldots \leq \Lambda_n$ and orthonormal eigenvectors $\{U_j\}_{j=1}^n$. Let $V$ be a vector and $\mu$ a real number such that for some $k$, $1 \leq k \leq n$,

$$(U_k, V) \geq 0, \quad \|V\| = 1, \quad \epsilon = AV - \mu V, \quad \nu = V^T AV.$$
Then if $\lambda_k$ is simple,
\[ \| U_k - V \|^2 \leq (\| \epsilon \|/\alpha_k)^2 + (\| \epsilon \|/\alpha_k)^4, \]
\[ |\nu - \lambda_k| \leq (\| \epsilon \|/\alpha_k) \times (1 - (\| \epsilon \|/\alpha_k)^2)^{-1}, \quad (13) \]
where $\alpha_k = \min_{l \neq k} |\lambda_l - \nu|$. If
\[ \lambda_{k-s-1} < \lambda_{k-s} \leq \ldots \leq \lambda_{k+t} < \lambda_{k+t+1}, \]
then
\[ \text{dist} \{ V, \text{Span} \{ \{U_j\}_{j=k-s}^{k+s} \} \} \leq \| \epsilon \|/d_k, \quad \min |\nu - \lambda_j| \leq \| \epsilon \|, \quad (14) \]
where $d_k = \min \{|\lambda_l - \nu|; k-s \leq m \leq k+t\}$.

We shall use Theorem W by applying it to $V = Q_n u_k/\|Q_n u_k\|$ and $\mu = \lambda_k$. We first recall that the Rayleigh-Ritz approximate eigenvalues and eigenvectors are those of the $n$-dimensional operator
\[ Q_n L_a Q_n = Q_n L_0 Q_n + Q_n B Q_n. \quad (15) \]

**Lemma 1.** For $1 \leq k \leq n$, let
\[ \epsilon_k = Q_n L_a Q_n u_k - \lambda_k Q_n u_k. \quad (16) \]
Then
\[ \| \epsilon_k \| \leq \| B(I - Q_n) u_k \|. \quad (17) \]

**Proof:** Since $Q_n$ is the orthogonal projection on the subspace spanned by the first $n$ eigenfunctions of $L_0$, we have $Q_n L_0 = L_0 Q_n$, $Q_n^2 = Q_n$. Thus, applying $Q_n$ twice to $L_0 u_k + B u_k = \lambda_k u_k$ gives
\[ Q_n L_0 Q_n u_k + Q_n B u_k = \lambda_k Q_n u_k. \quad (18) \]
Hence,
\[ \epsilon_k = Q_n B Q_n u_k - \lambda_k Q_n u_k, \quad (19) \]
and so
\[ \| \epsilon_k \| \leq \| Q_n \| \| B(I - Q_n) u_k \| = \| B(I - Q_n) u_k \|. \quad (20) \]
This proves (17).

Since
\[ \| Q_n u_k \|^2 = \| u_k \|^2 + \| (I - Q_n) u_k \|^2 = 1 + \| (I - Q_n) u_k \|^2, \quad (21) \]
(5) follows from (3), (16), (17), (21), and Theorem W (with $V = Q_n u_k/\|Q_n u_k\|$, $\mu = \lambda_k$, and $A = Q_n L Q_n$).

**Proof of Theorem 2:** The proof of Theorem 2 is similar to the proof of Theorem 1 in that Theorem W is used with $V = Q_n u_k/\|Q_n u_k\|$ and $\mu = \lambda_k$. The operator $A$, however, will be
\[ Q_n L_n Q_n = Q_n L_0 Q_n + Q_n B P_n Q_n, \quad (22) \]
as in reference 3, chapter 12, and not $Q_n L Q_n$. 

**Lemma 2.** For $1 \leq k \leq n$, let
\[ e_k = Q_nL_nQ_nu_k - \lambda_kQ_nu_k. \] (23)
Then
\[ ||e_k|| \leq ||B|| (||I - Q_n||u_k|| + (||B||/\gamma) ||I - Q_n||Bu_k||. \] (24)

**Proof:** From (18), we have
\[ e_k = Q_nBP_nQ_nu_k - Q_nBu_k = -Q_nB(I - P_n)u_k - Q_nBP_n(I - Q_n)u_k. \] (25)
Thus,
\[ ||e_k|| \leq ||B(I - P_n)u_k|| + ||BP_n|| ||I - Q_n||u_k||. \] (26)

Observe that if
\[ ||w|| = [w,w]^{1/2} = (Bw,w)^{1/2}, \] (27)
then from condition (c) of Theorem 2
\[ \gamma||w||^2 \leq ||w||^2 \leq ||B|| ||w||^2. \]
Therefore, since $BP_n$ is self-adjoint in $\mathcal{H}$,
\[ ||BP_n|| = \sup_{w \in \mathcal{H}} \frac{\langle BP_nw, w \rangle}{\langle w, w \rangle} \leq \sup_{w \in \mathcal{H}} \frac{\Vert P_nw, w \Vert}{\Vert w \Vert^2} \leq \sup_{w \in \mathcal{H}} \frac{\Vert w \Vert^2}{\Vert w \Vert^2} \leq ||B||. \] (28)

Also,
\[ ||B(I - P_n)u_k|| \leq ||B||^{1/2} ||I - P_n||u_k||. \] (29)
Since $B$ is self-adjoint in $\mathcal{H}$,
\[ ||B|| = \sup_{[w, w]} \frac{\langle Bw, w \rangle}{\langle w, w \rangle} = \sup_{[w, w]} \frac{\langle B^2w, w \rangle}{\langle w, w \rangle} \leq ||B||^2 \sup_{[w, w]} \frac{\Vert w \Vert^2}{\Vert w \Vert^2} \leq ||B||^2/\gamma. \] (30)
Now, $P_n$ is the orthogonal projection of $\mathcal{H}$ onto $\text{Span } [B^{-1}v_1]_{i=1}^\infty$, and
\[ u_k = \sum_{i=1}^\infty \langle Bu_k, v_i \rangle B^{-1}v_i. \] (31)
Thus
\[ ||I - P_n||u_k|| \leq \sum_{i=n+1}^\infty \langle Bu_k, v_i \rangle B^{-1}v_i|| \leq (1/\gamma)^{1/2} ||I - Q_n||Bu_k||. \] (32)
Combining (26), (28)–(30), and (32), we obtain (24).

Theorem 2 now follows from Wilkinson's theorem (with $V = Q_nu_k/||Q_nu_k||$, $\mu = \lambda_k$, and $A = Q_nL_nQ_n$), Lemma 2, (21), and
\[ (LQ_nu_k, Q_nu_k) = \lambda_k + (L(I - Q_n)u_k, (I - Q_n)u_k). \] (33)

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