A Singular Abstract Cauchy Problem*

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Abstract. In this note a singular abstract Cauchy problem is considered. A solution is obtained for this problem in terms of a regular abstract Cauchy problem. As an application we obtain a new solution of the initial value problem for a class of singular partial differential equations.

1. We shall utilize the notion of an abstract Cauchy problem, which was introduced by E. Hille,1 to discuss a singular abstract Cauchy problem of considerable interest in the applications. We show that if \( U \) is the infinitesimal generator of a strongly continuous group, then the singular abstract Cauchy problem under consideration has a solution. Furthermore, a representation formula for the solution is given.

2. We begin with a summary of semigroup(group) theory.2 Let \( X \) denote a complex Banach space with generic element \( f \). Let \( \| f \| \) denote the norm of \( f \). A family of linear bounded operators \( \{ S(t); t > 0 \} \) on \( X \) to itself is called a semigroup(group) if \( S(t_1 + t_2) = S(t_1)S(t_2) \) for all \( t_1, t_2 > 0 \) (for all \( t_1, t_2 \)). \( S(t) \) will be assumed to be continuous in the strong-operator topology for \( t > 0 \) (for all \( t \)). In this case

\[
w_0 = \lim_{t \to 0^+} t^{-1} \log \| S(t) \| < \infty \quad (w_0 = \lim_{t \to 0^+} |t|^{-1} \log \| S(t) \| < \infty).
\]

The semigroup(group) is said to be of class \((C_0)\) if \( \lim_{t \to 0^+} t\|S(t)\| = \lim_{t \to 0^+} S(t)f = f \)

for each \( f \in X \); it follows that given \( w > w_0 \), there exists an \( M > 0 \) such that \( \|S(t)\| \leq M \exp(wt), t > 0, \|S(t)\| \leq M \exp(w|t|), \) for all \( t \). For semigroups(groups) of \((C_0)\) \( \lim_{t \to 0^+} t^{-1}[S(t) - I]f = Af \) (\( \lim_{t \to 0^+} t^{-1}[S(t) - I]f = Af \)) exists for a set \( \mathcal{D}(A) \) dense in \( X \). \( A \) is called the infinitesimal generator of \( S(t) \); \( A \) is closed for semigroups(groups) of \( C_0 \).

3. The formulation of the singular abstract Cauchy problem given below is suggested by properties of groups of class \( C_0 \), and by the theory of the second-order abstract Cauchy problem.

SACP2: Given a complex Banach space \( X \) and a linear operator \( U \) with domain \( \mathcal{D}(U) \) and range \( \mathfrak{R}(U) \) in \( X \) and given two elements \( \phi_0, \phi_1 \) in \( X \), find a function \( Y(t) = Y(t; \phi_0, \phi_1) \) such that

(i) \( Y(t) \) is strongly absolutely continuous and twice continuously differentiable in each finite subinterval of \([0, \infty)\);
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(ii) for each $t > 0$, $Y(t) \in \mathcal{D}(U^2)$ and

$$Y(t) + (\rho/t) Y'(t) = U^2[Y(t)], \quad (3.1)$$

where $\rho$ is any real number, $-\infty < \rho < \infty$.

(iii) $\lim_{t \to 0^+} \|Y^{(k)}(t; \phi_0, \phi_1) - \phi_k\| = 0$, $k = 0, 1$.

For $\rho \neq 0$, equation (3.1) has variable coefficients; it depends on a parameter $\rho$. Since a coefficient of this equation becomes infinite for $t = 0$, we meet a singular abstract Cauchy problem when the initial data are given at $t = 0$. For $\rho = 0$, the equation reduces to the abstract Cauchy problem $\text{ACP}^2$, as was formulated by Phillips. The following result establishing a uniqueness and existence theorem for the $\text{ACP}^2$ was obtained by Hille.

THEOREM 1. If $U$ generates a group $[T(t); -\infty < t < \infty]$ strongly continuous at the origin with $T(0) = I$, then the corresponding $\text{ACP}^2$ has a unique solution for each $\phi_0 \in \mathcal{D}(U^2)$ and $\phi_1 \in \mathcal{D}(U) \cap \mathcal{R}(U)$, namely

$$y(t) = \frac{1}{2} [T(t) \phi_0 + z_1] + T(-t) (\phi_0 - z_1), \quad Uz_1 = \phi_1. \quad (3.2)$$

We note that this formula reduces to the solution of the equation for the vibrating string when $U = d/ds$.

We shall write $Y_\rho(t)$ instead of $Y(t)$ to indicate the dependence of $Y$ on the parameter $\rho$. One can show by a direct calculation that $Y_\rho(t)$ satisfies the following recurrence formulas

$$Y_\rho'(t) = tY_{\rho+2}(t), \quad (3.3)$$

and

$$Y_\rho(t) = t^{1-\rho}Y_{2-\rho}(t). \quad (3.4)$$

As a consequence of (3.3) we may assume that $\phi_1 = 0$. We shall let $Y(t)$ denote the unique solution of the $\text{ACP}^2$, namely

$$Y(t) = \frac{1}{2} [T(t) + T(-t)] \phi_0, \quad (3.5)$$

when $\phi_1 = 0$.

4. We are now able to state our main results.

THEOREM 2. Let $\rho > 0$ and let $U$ generate a group $[T(t); -\infty < t < \infty]$ strongly continuous at the origin with $T(0) = I$. Then the corresponding $\text{SACP}^2$ has a solution for each $\phi_0 \in \mathcal{D}(U^2)$ and $\phi_1 \equiv 0 \in \text{X}$, namely

$$Y_\rho(t) = \int_{-1}^t f(\rho; s) T(st) \psi_0 ds, \quad (4.1)$$

where $f(\rho; s) = C_\rho (t - s^{\rho/2} - 1)$ and $C_\rho = \Gamma((\rho + 1)/2)/\Gamma(\rho/2)\Gamma(1/2)$. We note that the integral in (4.1) is divergent for $\rho \leq 0$. The following result provides a solution for $\rho \in N' = \{x: x \leq 0, \text{and } x \text{ not a negative odd integer}\}$.

THEOREM 3. Let $\phi_0 \in \mathcal{D}(A^{m+1})$ where $m$ is the smallest positive integer such that $\alpha = 2m + \rho > 0$. Then

$$Y_\rho(t) = Ct^{1-\rho} \left( \frac{d}{dt} \int_{-1}^1 f(\rho; s) T(st) \psi_0 ds \right) \quad (4.2)$$
is a solution of SACP² valid for ρ∈N'. Here C = 1/[(ρ + 1)(ρ + 2) ... (α + 1)]. We remark that CCρ is infinite when ρ is an odd negative integer.

The proofs of the above theorems employ the following lemma which we state without proof.

**Lemma 1.** Let T(s) be a strongly continuous group with infinitesimal generator U and let ϕ be in the domain of Un. Then T(s)ϕ is n times continuously differentiable with respect to s, and

\[
(d/ds)^k T(s)\phi = U^k T(s)\phi
\]

for 0 ≤ k ≤ n.

We now prove Theorem 2. To show that (4.1) satisfies (3.1), we compute:

\[
Y_\rho'(t) = \int_{-1}^{1} f(\rho; s)(d/dt)(T(st)\phi_0)ds
\]

= \int_{-1}^{1} sf(\rho; s)UT(st)\phi_0 ds (by Lemma 1);

\[
y''(t) = \int_{-1}^{1} tf(\rho; s)(d^2/dt^2)(T(st)\phi_0)ds
\]

= \int_{-1}^{1} tf(\rho; s)s^2U^2T(st)\phi_0 ds (by Lemma 1)

= \{s^2f(\rho; s)UT(st)\phi_0\}|_{-1}^{1} - \int_{-1}^{1} \{(d/ds)(s^2f(\rho; s))UT(st)\phi_0\}ds

(by Lemma 1, and integration by parts);

\[
tU^2[Y_\rho(t)] = tU^2\int_{-1}^{1} f(\rho; s)T(st)\phi_0 ds
\]

= \int_{-1}^{1} tf(\rho; s)U^2T(st)\phi_0 ds (U^2 is closed)

= \{f(\rho; s)UT(st)\phi_0\}|_{-1}^{1} - \int_{-1}^{1} \{(d/ds)(f(\rho; s))UT(st)\phi_0\}ds

(by Lemma 1, and integration by parts).

Thus,

\[
ty''(t) + \rho y''(t) - tU^2[Y_\rho(t)] = C_\rho \{ - (1 - s^2)^{\rho/2}UT(st)\phi_0 \}|_{-1}^{1} + \int_{-1}^{1} \{(d/ds) ((1 - s^2)^{\rho/2} + \rho s(1 - s^2)^{\rho/2 - 1}) UT(st)\phi_0 ds \}
\]

= 0 for ρ > 0.

We show now that the initial conditions are satisfied. It is now advantageous to write (4.1) in the following form

\[
Y_\rho(t) = \int_{0}^{t} f(\rho; s)|T(st) + T(-st)|\phi_0 ds = 2\int_{0}^{t} f(\rho; s)Y(st)ds.
\]

Now let A_k = 2\int_{0}^{t} |f_k(s)|ds (k = 0, 1) where f_0(s) = f(\rho; s) and f_1(s) = sf(\rho; s).

For every ε_k > 0, there exist δ_k > 0 such that

\[
||Y^{(k)}(t) - \phi_k|| \leq \epsilon_k/A_k \quad (k = 0, 1)
\]

whenever 0 < t < δ_k. This statement is a consequence of Y(t) being the solution of the ACP². We recall that φ_1 = 0. Thus for 0 < t < δ_k, we have

\[
||Y^{(k)}(t) - \phi_k|| \leq 2\int_{0}^{t} |f_k(s)| ||Y^{(k)}(st) - \phi_k|| ds \leq 2(\epsilon_k/A_k) \int_{0}^{t} |f_k(s)|ds \leq \epsilon_k, \quad k = 0, 1.
\]

This completes the proof of Theorem 2.
A device of A. Weinstein is used to prove Theorem 3. In our proof, we use Theorem 2 along with the repeated application of the recurrence formulas (3.3) and (3.4). Let \( m \) be the smallest positive integer such that \( \alpha = \rho + 2m > 0 \). Theorem 2 states that equation (3.1), where now \( \rho \) has been replaced by \( \alpha \), has a solution satisfying the initial conditions

\[
\lim_{t \to 0^+} \left\| Y_a^{(k)}(t) - \varphi_k \right\| = 0, \quad k = 0,1;
\]

where \( \varphi_0 = C\varphi_0 \) and \( \varphi_1 = 0 \). By (3.4) we have

\[
\ell^{a-1}Y_a(t) = Y_{2-a}(t)
\]

and applying (3.3) \( m \) times, we obtain

\[
Y_{2-\rho}(t) = \left( \frac{1}{t} \frac{d}{dt} \right)^m (\ell^{a-1}Y_a(t)).
\]

Now upon employing (3.4) once more, we obtain the expression

\[
Y_{\rho}(t) = t^{1-\rho} \left( \frac{1}{t} \frac{d}{dt} \right)^m (\ell^{a-1}Y_a(t)). \tag{4.4}
\]

It is clear from the construction, which starts from \( Y_a \), that \( Y_{\rho} \) is a solution of the abstract operator equation in (3.1) if \( Y_a(t) \) has \( m \) derivatives. It is sufficient that \( \varphi_0 \) be contained in the domain of \( U^{m+2} \). Upon carrying out the differentiation in (4.4) we obtain an expression of the form

\[
Y_{\rho} = (\rho + 1)(\rho + 2) \ldots (\alpha - 1)Y_a(t) + BtY_a'(t) + 0(t^2), \tag{4.5}
\]

where \( B \) is a constant. It follows from (4.5) that the initial conditions are satisfied and this completes the proof.

Upon setting \( \rho = 0 \) in (4.2) we obtain the following expression

\[
Y_0(t) = \frac{1}{\sqrt{2\pi}} \int \left( t \int_{-1}^1 T(st) \phi ds \right) dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int [F(t) - F(-t)] dt
\]

(where \( F(t) = \int T(s) \phi ds \))

\[
= \frac{1}{\sqrt{2\pi}} [T(t) + T(-t)] \phi
\]

\[
= Y(t),
\]

the representation of the solution of the ACP\(^2\) satisfying the initial conditions \( \varphi_0 = \phi \) and \( \varphi_1 = 0 \).

5. For \( \rho \geq 0 \) the solutions of the SACP\(^2\) depend continuously on the initial data. This can be seen as follows. By Theorem 1, for each \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
\left\| Y(t; f_1) - Y(t; f_2) \right\| < \epsilon/A
\]

whenever \( \|f_1 - f_2\| < \delta \). Here \( A = 2 \int_{0}^{\rho} |f(\rho; s)| ds \). Thus for \( \|f_1 - f_2\| < \delta \), we have
\[ \|Y_{\rho}(t; f_1) - Y_{\rho}(t; f_2)\| \leq 2 \int_0^1 |f(\rho; s)| \|Y(st; f_1) - Y(st; f_2)\| ds \]

\[ < \epsilon.\]

The case \( \rho = 0 \) is covered by Theorem 1. We have

**Theorem 4.** For \( \rho \geq 0 \), the solution operator of the SACP\(^2\) is continuous.

6. In this section we discuss briefly uniqueness questions for the SACP\(^2\).

One can easily verify for \( \rho < 0 \) that \( W_{\rho} = Y_{\rho} + t^{4-\rho}Y_{2-\rho} \), where \( Y_{2-\rho} \) is a sufficiently smooth solution of (3.1) with \( \rho \) replaced by \( 2 - \rho \), is a solution of the SACP\(^2\) whenever \( Y_{\rho} \) is a solution. Thus the solutions are not unique when \( \rho < 0 \).

The uniqueness of the solution for \( \rho > 0 \) has been established for a special case of the SACP\(^2\). Here we shall outline a procedure for demonstrating the uniqueness of the solution of the SACP\(^2\). One notes that (4.3) may be expressed in terms of the Riemann-Liouville fractional integration operator

\[ I^\alpha[f] = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1}f(t)dt, \]

namely

\[ Y_{\rho}(t^{1/\rho}) = \frac{1}{\rho} C_{\rho}^{1/\rho-\rho/2} \int_0^t (t - s)^{\rho/2-1}s^{-1/\rho}Y(s^{1/\rho})ds, \rho > 0. \]

Now if \( S \) and \( S_{\rho} \) denote the solution sets of the ACP\(^1\) and the SACP\(^3\), respectively, then (6.2) defines a one-one mapping from \( S \) into \( S_{\rho} \). This follows from a theorem of Titchmarsh which states that if \( f \) and \( g \) are continuous functions and \( \int_0^1 f(t - x)g(x)dx = 0 \), then \( f = 0 \), or \( g = 0 \). One inverts the operator in (6.2) and obtains a mapping from \( S_{\rho} \) into \( S \). The uniqueness of the solution of the SACP\(^2\) follows now from the uniqueness of the solution of the ACP\(^1\). A future paper will contain the details of the above procedure.

7. We illustrate the above theory with an application to the classical Cauchy problem for the Euler-Poisson-Darboux equation

\[ V_{tt} + (\rho/t)V_t = \Delta_x V, \]

where \( \Delta_x = \sum_{i=1}^n \partial^2 / \partial x_i^2 \), the \( n \)-dimensional Laplacian operator. Let \( X = L_2(\mathbb{R}^n) \), the space of square integrable measurable functions on \( \mathbb{R}^n \). We are to find a solution of (7.1) satisfying the initial conditions \( V(x, 0) = f(x) \), and \( V_t(x, 0) = 0 \), where \( x = (x_1, x_2, \ldots, x_n)e\mathbb{R}^n \) and \( f \in L_2 \). It is well known that the square root of the Laplacian operator generates a strongly continuous group on \( L_2 \) with respect to the usual norm. Let us denote by \( \tilde{\sigma}[f](\xi) \) and \( \tilde{\sigma}^{-1}[f](\xi) \) respectively, the Fourier transform and the inverse Fourier transform of \( f \). Upon applying Theorem 2, we obtain

\[ V(x, t) = \int_0^1 f(\rho; s) [\tilde{\sigma}^{-1}[\tilde{\sigma}[f](\xi) \cos(|\xi||st|)] ds, \]

where \( ||\xi|| = (\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^{1/2} \), as the solution of this problem, valid for \( \rho > 0 \). We may use Theorem 3 to obtain a solution for \( \rho e^{N^2} \).

For other solutions of the Cauchy problem for the Euler-Poisson-Darboux equation, see Weinstein,\(^4\) and Diaz and Weinberger.\(^6\) For a discussion of the exceptional case \( \rho = -1, -3, -5, \ldots \), see Blum.\(^6\)

Of course, our $L_2$ solution of the Euler-Poisson-Darboux equation could also be obtained by a direct application of Fourier transformations in the spatial variables in (7.1) in the usual manner. On the other hand, our general formula (4.1) is also valid when $\Delta_n$ is replaced by any differential operator $P(x, D)$ whose square root generates a strongly continuous group on some function space $X$.

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\footnote{Hille, E., "Une généralisation du problème de Cauchy," Ann. de l'Inst. Fourier (Grenoble), 4, 31–48 (1953).}
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