A Class of Neutral Equations with the Fixed Point Property

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Abstract. For a neutral functional differential equation with a stable operator, D, it is shown that the solution operator is the sum of a contraction and a completely continuous operator.

\[ D(\varphi) = \int_{-r}^{0} [d\eta(\theta)] \varphi(\theta) \]
\[ \det[\eta(0) - \eta(0-)] \neq 0 \]
\[ |\int_{-s}^{0} [d\eta(\theta)] \varphi(\theta)| \leq \gamma(s)|\varphi| \]

for all \( \varphi \in C, s \geq 0 \), where \( \eta(\theta) \) is an \( n \times n \) matrix function of bounded variation in \( \theta \) and \( \gamma(s) \) is a continuous scalar function for \( s \geq 0, \gamma(0) = 0 \).

For a function \( x \) defined on any interval \([\sigma - r, b], b \geq \sigma\), let \( x_t \) be a function on \([\sigma - r, 0]\) defined by \( x_t(\theta) = x(t + \theta), -r \leq \theta \leq 0 \). If \( f: [0, \infty) \times C \to E^n \) is continuous, a neutral functional differential is a relation

\[ \frac{d}{dt} D(x_t) = f(t, x_t). \]  

A function \( x \) defined on an interval \([\sigma - r, \sigma + A], A > 0\) is said to be a solution of (2) if \( x \) is continuous on this interval, \( D(x_t) \) is continuously differentiable and satisfies (2) on \((\sigma, \sigma + A)\). We assume that for any \((\sigma, \varphi) \in [0, \infty) \times C\), there is a unique solution \( x(\sigma, \varphi) \) of (2) with \( x_\sigma(\sigma, \varphi) = \varphi \), defined on \([\sigma - r, \infty)\) and \( x(\sigma, \varphi)(t) \) is continuous in \((\sigma, \varphi, t)\). Let the solution operator \( T(t, \sigma): C \to C \) for \( t \geq \sigma \) be defined by

\[ T(t, \sigma) \varphi = x_t(\sigma, \varphi). \]

For a certain class of operators \( D \), we will prove there is a \( \beta > 0 \) such that the operator \( T(t, \sigma) \) for \( t \geq \sigma + \beta \) is the sum of a contraction operator and a completely continuous operator. Therefore, the fixed point property holds for \( t \geq \sigma + \beta \).

The operator \( D \) is said to be stable \([1]\) if there are \( K \geq 1, \alpha > 0 \) such that the solution \( y(\varphi) \) of

\[ D(y_t) = D(\varphi), y_0 = \varphi, \]

satisfies

\[ |y_t(\varphi)| \leq Ke^{-\alpha t}|\varphi| + K|D(\varphi)|, t \geq 0, \varphi \in C. \]
From [2], there is an \( n \times n \) matrix \( B(t) \) continuous in \( t \) from the left, of bounded variation in \( t \) on compact subsets of \( [0, \infty) \), \( B(t) = 0 \) for \( t \leq 0 \) such that the solution \( x = x(\sigma, \varphi) \) of (2) satisfies

\[
x(t) = y(t - \sigma) + \int_{t-s}^{t} B(t - s)f(s, x_s)ds
\]

where \( y = y(\varphi) \) is the solution of (4).

**Theorem.** If \( T(t, \sigma) \) takes bounded sets of \( C \) into bounded sets of \( C \), \( f \) takes bounded sets of \( [0, \infty) \times C \) into bounded sets of \( \mathbb{R}^n \) and \( D \) is stable, then there is a \( \beta > 0 \) such that

\[
T(t, \sigma) = T_1(t, \sigma) + T_2(t, \sigma)
\]

where \( T_1(t, \sigma) \) is a contraction for \( t \geq \sigma + \beta \) and \( T_2(t, \sigma) \) is completely continuous for \( t > \sigma + \beta \).

**Proof:** From [3], there are \( n \) functions \( \varphi_1, \ldots, \varphi_n \) in \( C \) such that if \( \Phi = (\varphi_1, \ldots, \varphi_n) \), then \( D(y_i(\Phi))) = I, t \geq 0 \). If

\[
x(t) = y(\Phi D(\varphi))(t - \sigma) + z(t)
\]

then \( D(z_i) = 0 \) and from (6)

\[
z(t) = y(z_i)(t - \sigma) + \int_{t-s}^{t} B(t - s)f(s, z_s) + y_{z,p}(\Phi D(\varphi))
\]

From the definition of \( z_\sigma \) and the continuity of \( D \), there is a constant \( N \) such that \( |D(z_\sigma)| \leq N|\varphi| \). Since \( D(z_\sigma) = 0 \), it follows from (5) that

\[
|y_i(z_\sigma)| \leq KN e^{-\alpha t}|\varphi|, t \geq 0.
\]

The linearity of \( y_i \) in \( z_\sigma \) and the linearity of \( z_\sigma \) in \( \varphi \) implies that \( y_i(z_\sigma) \) considered as a function of \( \varphi \) is a contraction mapping for \( t \geq \beta \), \( KN \exp(-\alpha t) < 1 \).

Let

\[
w(t) = \int_{t-s}^{t} B(t-s)f(s, x_s)ds.
\]

Suppose \( U \) is a bounded set of \( C \). By hypothesis \( T(s, \sigma)U \) is bounded and \( f(s, T(s, \sigma)U) \) is bounded. Ascoli’s theorem implies that \( w \), maps \( U \) in a pre-compact set of \( C \) for \( t \geq r \).

Therefore, \( z_\sigma \), considered as mapping on \( C \) is the sum of a map which is a contraction for \( t \geq \sigma + \beta \) and a map which is completely continuous for \( t \geq \sigma + r \). Since the range of the map \( y_i(\Phi D(\cdot)) \) is finite dimensional, it is compact. Finally, (7) implies \( T(t, \sigma) \) satisfies the properties stated in the theorem and the proof is complete.

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