Embedding Trees in the Rationals

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Abstract. An example is presented of a simple algebraic statement whose truth cannot be decided within the framework of ordinary mathematics, i.e., the statement is independent of the usual axiomatizations of set theory. The statement asserts that every tree-like ordering of power equal to or less than the first uncountable cardinal can be embedded homomorphically into the rationals.

In this paper, we show (see also refs. 1–3) that the following assertion A, which implies Souslin's hypothesis, is independent of the axioms of set theory (more precisely, both A and its negation are relatively consistent with ZFC, the Zermelo-Fraenkel axioms plus the axiom of choice).

A. If $T$ is a tree-like partial order in which every chain is countable and if $T$ has cardinality $\aleph_1$, then there is an embedding $f: T \rightarrow Q$ (Q the rational numbers) such that $x < y$ implies $f(x) < f(y)$.

The following fragment of A already implies Souslin's hypothesis.

B. Every Aronszajn tree can be embedded in the rationals.

Questions about embeddings of trees into linear orders have been discussed before (see Kurepa4 and Johnston5).

Definition 1: Let $P$ be a partial order.

(a) $P$ is tree-like iff for each $q \in P$, $P_q = \{ x \in P: x \leq q \}$ is linearly ordered.

(b) $x, y \in P$ are compatible if there is $r \in P$ with $x, y \leq r$, otherwise $x, y$ are incompatible.

(c) $C \subseteq P$ is an antichain iff for all $x, y \in C$, if $x \neq y$ then $x, y$ are incompatible.

(d) $C \subseteq P$ is a chain iff $C$ is linearly ordered.

(e) If $T$ is a tree, the order $l(t)$ of $t \in T$ is the order type of $\{ s \in T: s < t \}$.

(f) An Aronszajn tree is a tree with no uncountable chains in which each level $\{ t: l(t) = \alpha \}$ ($\alpha \in \omega_1$) is countable, such that above each point $t$ there are points $s$ of every countable order $\geq l(t)$.

(g) A Souslin tree is an Aronszajn tree which contains no uncountable antichains.

(h) $P$ is embeddable in the rationals iff there is a function $f: P \rightarrow Q$ preserving strict order, i.e., such that $x < y$ implies $f(x) < f(y)$. Note that $f$ is not required to be one-one.

We will write $\text{card } x$ for the cardinality of $x$. 1748
Notice that if $P$ is tree-like then $C \subseteq P$ is an antichain iff for all $x, y \in C$, neither $x < y$ nor $y < x$.

Souslin's hypothesis (SH) asserts that there are no Souslin trees.

It is easy to see that if $T$ is an Aronszajn tree then $\text{card } T = \aleph_1$. Hence A implies B. Next we assert that B implies Souslin's hypothesis. Suppose $T$ is a Souslin tree and $f: T \to Q$ is an embedding. Then $T = \bigcup \{T_r: r \in Q\}$, where $T_r = \{p \in T: f(p) = r\}$, so some $T_r$ is uncountable. But it is easy to see that each $T_r$ is an antichain, contradicting the fact that $T$ is Souslin. Since the negation of Souslin's hypothesis is known to be consistent (see Jech and Tennenbaum⁷), it follows that the negation of B (and hence the negation of A) is consistent.

In Theorem 4 below, we show that A follows from Martin's Axiom together with $\aleph_1 < 2^\aleph_0$. Since Martin's Axiom is known to be consistent with $\aleph_1 < 2^\aleph_0$ (see Solovay and Tennenbaum⁴), A is consistent, and hence A is independent of the axioms of set theory.

Baumgartner first showed the consistency of A for trees; Malitz and Reinhardt independently of Baumgartner (but not of each other) showed this and the consistency of A. Later Baumgartner observed that A follows from the special case for trees. Several other results concerning embeddings and decompositions of trees have been obtained by Baumgartner and will appear in another paper. (Some of these results are announced in ref. 2.) Laver has shown (in ZFC) that there is a tree of power $2^\aleph_0$ embeddable in the reals but not in the rationals and such that each uncountable subset contains an uncountable antichain. (Subsequently Galvin found an example of such a tree which cannot be embedded in the reals. Proofs of these results may be found in ref. 1.) Thus the negation of A is provable in ZFC + $2^\aleph_0 = \aleph_1$. This, together with Jensen's recent result (unpublished) that SH is consistent with $2^\aleph_0 = \aleph_1$ shows that A does not follow from SH. The question whether B follows from SH is open.

The Laver example also shows that $\aleph_1$ cannot be replaced by $2^\aleph_0$ in A. The following partial order (due to Sierpiński) shows that the condition in A that $T$ be tree-like cannot be weakened to $T$ an arbitrary partial order. Let $P = \{r_\alpha: \alpha \in \omega_1\}$ be an uncountable set of distinct reals. Put $r_\alpha < r_\beta$ iff $r_\alpha < r_\beta$ in the usual ordering of the reals, and also $\alpha < \beta$. It is easy to see that every set of pairwise incomparable elements of $P$ is countable and hence, as in the proof that B implies SH, $P$ cannot be embedded in the rationals.

**Definition 2:** (a) Let $P$ be a partial order. A set $D$ is called $P$-dense if $D \subseteq P$ and for all $p \in P$ there is $q \geq p$ such that $q \in D$.

(b) A subset $G$ of $P$ is called $P$-generic over $M$ iff the following three conditions hold.

(i) if $p, q \in G$ then there is $r \in G$ with $p, q \leq r$

(ii) if $q \in G$ and $p \leq q$ then $p \in G$

(iii) if $D \subseteq M$ is $P$-dense, then $G \cap D \neq 0$.

**Martin's Axiom.** If $P$ is a partial order in which every antichain is countable, and if $M$ is any set of cardinality $< 2^\aleph_0$, then there is a set $G \subseteq P$ such that $G$ is $P$-generic over $M$.

For a general discussion of Martin's Axiom and its consequences, see Martin and Solovay⁷.
If \( T \) is a partial order and \( p \subseteq T \times Q \) is a finite function, then we say that \( p \) is a partial embedding if \( x'y \in \text{dom } p \) and \( x < y \) implies \( p(x) < p(y) \). With any partial order \( T \) we associate the set \( P \) of partial embeddings, ordered by inclusion. If \( T \) is the order mentioned in \( A \), then it is the partial order \( P \) to which we apply Martin's Axiom. The only difficult point is to see that the antichain condition in Martin's Axiom is satisfied. This is taken care of by Theorem 3. We first give the proof that Theorem 4 follows from Theorem 3.

**Theorem 3.** Let \( T \) be a tree-like partial order which contains no uncountable chains. Let \( P \) be the partial order of partial embeddings of \( T \) described above. Then every antichain contained in \( P \) is countable.

**Theorem 4.** Assume Martin's Axiom. Let \( T \) be any tree-like partial order in which every chain is countable, and card \( T < 2^{\aleph_0} \). Then there is a function \( f:T \rightarrow Q \) such that \( x < y \) implies \( f(x) < f(y) \).

**Proof of 4:** Let \( P \) be the set of partial embeddings of \( T \). For each \( x \in T \) let \( D_x = \{ p \in P : x \in \text{dom } p \} \). We claim each \( D_x \) is \( P \)-dense. Suppose \( p \in P \) and \( x \notin \text{dom } p \). Let \( X = \{ p(y) : y \in \text{dom } p \text{ and } y < x \} \) and let \( Y = \{ p(y) : y \in \text{dom } p \text{ and } x < y \} \). Since \( p \in P \) it follows that if \( r_1 \in X \) and \( r_2 \in Y \) then \( r_1 < r_2 \).

Since \( Q \) is densely ordered there exists \( r \in Q \) such that for all \( r_1 \in X, r_1 < r \) and for all \( r_2 \in Y, r < r_2 \). Then it is easy to see that \( p \cup \{ (x,r) \} \in D_x \), and \( p \cup \{ (x,r) \} \) extends \( p \). Hence \( D_x \) is \( P \)-dense.

Let \( M = \{ D_x : x \in T \} \). Clearly card \( M = \text{card } T < 2^{\aleph_0} \). By Theorem 3, \( P \) has no uncountable antichains, so we may apply Martin's Axiom to obtain a set \( G \) which is \( P \)-generic over \( M \). We assert that \( \cup G \) is the desired embedding. It is easy to see that \( \cup G \) is a function since \( G \) satisfies condition (i) of the definition of generic. For each \( x \in T \) we have \( G \cap D_x \neq 0 \), so \( x \in \text{dom } \cup G \) and hence \( \text{dom } \cup G = T \). If \( x, y \in T \) and \( x < y \) then there are \( p,q \in G \) such that \( x \in \text{dom } p \) and \( y \in \text{dom } q \). But then \( \cup G(x) = p(x) < q(y) = \cup G(y) \) since \( p \) and \( q \) must be compatible. This completes the proof of Theorem 4.

**Proof of 3:** We shall prove by induction on \( n \) that

* If \( B \subseteq P \) is an antichain and card \( p = n \) for all \( p \in B \), then \( B \) is countable.

We assume now that \( * \) is true for \( m < n \). We proceed by contradiction. Let \( B \subseteq P \) be an antichain with card \( p = n \) for all \( p \in B \), and suppose \( B \) is uncountable.

We split the induction step of the proof into a number of steps; throughout these steps, \( B \) and \( n \) are fixed as above. Some notation will be useful. If card \( p = m \), then \( p = \{ p_0, \ldots, p_{m-1} \} \), where each \( p_i \) is an ordered pair which we shall write as \( p_i = (\hat{p}(i), p'(i)) \). Thus \( \hat{p} \) enumerates the domain of \( p \), \( p' \) the range of \( p \), and \( p(\hat{p}(i)) = p'(i) \). So that \( \hat{p} \) will be uniquely determined by \( p \), we linearly order \( T \) (in any way) and require that \( \hat{p} \) enumerate dom \( p \) in increasing order.

For any \( D \subseteq P \) we introduce the notation \( D^i = \{ \hat{p}(i) : p \in D \} \). If \( p \in P \), card \( p = n \) and \( i < n \), we write \( p^i = p - \{ p_i \} = \{ \hat{p}(j), p'(j) : j < n \text{ and } j \neq i \} \); we have \( p^i \subseteq P \) and card \( p^i = n - 1 \).

Notice that if \( x \in T \) then \( \{ y \in T : y < x \} \) is a chain since \( T \) is tree-like, and hence must be countable. This observation will be used frequently in the rest of the proof.
**Step 1:** Let \( C \subseteq P \) be uncountable. Then there is some \( D \subseteq C, D \) uncountable, such that

(a) for all \( p,q \in D, p' = q' \)

(b) if \( C' \) is countable, then for all \( p,q \in D, \tilde{p}(i) = \tilde{q}(i) \).

**Proof:** \( \{p': p \in C\} \) is countable, and to say that \( C' \) is countable is to say that \( \{\tilde{p}(i): p \in C\} \) is countable.

**Step 2:** Suppose that \( C \subseteq B \) is uncountable. Then for each \( i < n, C_i \) is uncountable.

**Proof:** Suppose \( C_i \) is countable, and \( i < n \). Then step 1 allows us to choose \( D \subseteq C, D \) uncountable, and \( x \in T \) such that for all \( p,q \in D, p' = q' \) and \( \tilde{p}(i) = z. \) Evidently \( D \) is still an antichain. Let \( E = \{p': p \in D\} \). Now we claim that if \( p,q \in D \) then \( p' \) is incompatible with \( q' \). From the claim it follows that the function \( p \mapsto p' \) is one-one, so card \( E = \text{card D} \) and \( E \) is an uncountable antichain with card \( p' = n - 1 \) for all \( p' \in E \). This contradicts the induction hypothesis.

To prove the claim, first observe that if \( p,q \) are incompatible, then there are \( x \in \text{dom} p, y \in \text{dom} q \) so that either

(i) \( x = y \) and \( p(x) \neq q(y) \),

(ii) \( x < y \) and \( p(x) \geq q(y) \) or

(iii) \( y < x \) and \( p(x) \leq q(y) \).

Now suppose that \( p,q \in D \). In case (i), if \( x = y \neq z \), then \( x \in \text{dom} p', y \in \text{dom} q' \) and \( p',q' \) are clearly incompatible. If \( x = y = z \), then case (i) does not arise because \( p'(i) = q'(i) \). In case (ii), if \( x,y \neq z \) it is again clear that \( p',q' \) are incompatible. If \( x = z \), then we have (a) \( z < y \) and \( p(z) \geq q(y) \). But since \( p' = q' \) and \( z = \tilde{p}(i) = \tilde{q}(i), p(z) = p'(i) = q'(i) = q(z) \), and we have (b) \( q(z) \geq q(y) \). Now (a) and (b) contradict \( q \in P \), so \( x = z \) cannot arise in case (ii) either. Similarly, \( y = z \) cannot arise. Case (iii) is similar to case (ii), so the claim is proved.

Let \( U(C) \) denote the following condition on a subset \( C \) of \( B \):

\[ U(C): \text{ For every } i < n \text{ and every } x \in C_i \text{ there is a unique } p \in C \text{ such that } x = \tilde{p}(i). \]

**Step 3:** Suppose that \( C \subseteq B \) is uncountable. Then there is an uncountable \( D \subseteq C \) such that \( U(D) \) holds.

**Proof:** By induction on \( i \), suppose that \( E \subseteq C \) is uncountable and for all \( j < i \), and all \( x \in E^i \), there is a unique \( p \in E \) with \( x = \tilde{p}(j) \). By step 2, \( E^i \) is uncountable. For each \( x \in E^i \) we choose exactly one \( p_x \in E \) so that \( \tilde{p}_x(i) = x \), and we set \( D = \{p_x: x \in E^i\} \). Then \( D \) is uncountable and satisfies the uniqueness condition for \( j \leq i \).

**Step 4:** Suppose that \( C \subseteq B \) is uncountable. Then there is an uncountable \( D \subseteq C \) such that for each \( i < n \) either \( D_i \) is an uncountable antichain or \( D_i \) contains no uncountable antichain.

**Proof:** We construct a sequence \( C_0 \supseteq C_1 \supseteq \ldots \supseteq C_n \) of uncountable subsets of \( C \) as follows: Let \( C_0 = C \). Suppose we have constructed \( C_i \). If \( S \subseteq C_i \) is an uncountable antichain, then for each \( x \in S \) choose \( p_x \in C_i \) so that \( \tilde{p}_x(i) = x \), and let \( C_{i+1} = \{p_x: x \in S\} \). If \( C_{i+1} \) contains no uncountable antichains, let
$C_{t+1} = C_t$. Then $C_{t+1}$ is uncountable, and either $(C_{t+1})^i$ is an uncountable antichain or $(C_{t+1})^i$ contains no uncountable antichains. Let $D = C_n$. It is clear that $D$ satisfies the assertion in step 4.

**Step 5:** Suppose $R \subseteq T$ and $R = R_0 \cup R_1 \cup \ldots \cup R_{m-1}$, where each $R_j$ is uncountable. Suppose further that $R$ contains no uncountable antichains. Then there exist distinct $x_0, x_1, \ldots, x_{m-1} \in R$ such that \( \{ x_j : j < m \} \) is an antichain and for each $j < m$, \( \{ y \in R_j : x_j \leq y \} \) is uncountable.

**Proof:** We proceed by induction on $m$. Suppose $m = 1$. Let $S \subseteq R$ be a maximal antichain. Then $S$ is countable and every member of $R$ is comparable with some member of $S$. Moreover, for each $x \in S$ the set \( \{ y \in R : y \leq x \} \) is countable. Therefore, for some $x_0 \in S$, \( \{ y \in R : x_0 \leq y \} \) is uncountable.

Suppose $m = 2$. Let $S' = \{ x \in R_0 : \{ y \in R_0 : x \leq y \} \) is uncountable. Then $S'$ is uncountable, for if not then $R_0 - S'$ is uncountable and by the case just proved for $m = 1$ there would exist $y \in R_0 - S'$ such that \( \{ z \in R_0 - S' : y \leq z \} \) is uncountable, and then $y \in S'$, a contradiction. Since $S'$ is uncountable, it cannot be a chain. Hence we may choose incomparable $y_0, y_1 \in S'$. If \( \{ z \in R_i : y_l \leq z \} \) is uncountable, then, letting $x_0 = y_0$ and $x_1 = y_1$, we are done. If not, then $R'_1 = \{ z \in R_1 : z \) is incomparable with $y_1 \} \) is uncountable, and by step 5 for $m = 1$ there is $y \in R'_1$ so that \( \{ z \in R'_1 : y \leq z \} \) is uncountable. But then $y$ and $y_1$ are incomparable, so we let $x_0 = y_0$ and $x_1 = y_1$.

Suppose $m > 2$. Let $R' = R_0 \cup R_1 \cup \ldots \cup R_{m-2}$. By inductive hypothesis there exist mutually incomparable $x_0', x_1', \ldots, x_{m-2}' \in R'$ so that \( \{ y \in R_j : x_j' < y \} \) is uncountable for each $j < m - 1$. Suppose that for some $j < m - 1 \{ y \in R_{m-1} : x_j' \leq y \} \) is uncountable. Then, applying step 5 for $m = 2$ to \( \{ y \in R_j : x_j' \leq y \} \cup \{ y \in R_{m-1} : x_j' \leq y \} \) we obtain incomparable $y_0, y_1 \geq x_j'$ so that \( \{ y \in R_j : y_0 \leq y \} \) and \( \{ y \in R_{m-1} : y_1 \leq y \} \) are both uncountable. Let $x_1 = x_i'$ for all $i < m - 1, i \neq j$; let $x_j = y_0$ and $x_{m-1} = y_1$. Since $T$ is tree-like \( x_j : j < m \) is an antichain. The second condition is clear. If the case above does not occur, then for all $j < m - 1 \{ y \in R_{m-1} : x_j' \leq y \} \) is countable. Then $R'' = \{ y \in R_{m-1} : y \) is incomparable with each $x_j' \} \) is uncountable, and by step 5 for $m = 1$ there is $x_{m-1} \in R''$ so that \( \{ y \in R'' : x_{m-1} \leq y \} \) is uncountable. Let $x_j = x_j'$ for all $j < m - 1$. This completes the proof.

**Step 6:** If $C \subseteq B$ is uncountable, then for every natural number $m$ there is a set $D \subseteq C$ such that card $D \geq m$ and for each $i < n D^i$ is an antichain.

**Proof:** By steps 3 and 4 we may assume that $U(C)$ holds and that for each $i < n$ either $C^i$ is an uncountable antichain or else $C^i$ contains no uncountable antichains. (By step 2, of course, each $C^i$ is uncountable.) Let $H = \{ i < n : C^i \) contains no uncountable antichains \}. If $H$ is empty then it is clear that $D = C$ works. Suppose card $H = k > 0$. Let $h_0, \ldots, h_{k-1}$ enumerate $H$ and let $m$ be fixed. It is now a simple matter to use step 5 repeatedly to construct simultaneously a sequence \( \langle x_i : i < k, j < m \rangle \) of elements of $T$ and a sequence \( \langle C_{ij} : i \leq k, j < m \rangle \) of subsets of $C$ such that the following conditions are satisfied:

1. $C_{ij} = C$ for each $j < m$
2. For each $i < k, x_{i,0}, \ldots, x_{i,m-1}$ are incomparable members of $T$ such that for each $j < m$, \( \{ y \in (C_{ij})^j : x_{ij} \leq y \} \) is uncountable, and hence \( \{ p \in C_{ij} : x_{ij} \leq \hat{p}(h_i) \} \) is uncountable.
(3) For each i with 0 < i ≤ k, \( C_{ij} = \{p \in C_{i-1,j}: x_{i-1,j} \leq \hat{p}(h_{i-1})\} \).

Finally, for each j < m, choose \( p_j \in C_{kj} \). We claim \( D = \{p_j: j < m\} \) works. Suppose \( j_1 < j_2 < m \) and \( i < n \). We must show that \( \hat{p}_{n}(i) \) and \( \hat{p}_{n}(i) \) are incomparable. If \( i \notin H \) then this follows from the fact that \( U(C) \) holds and \( C_i \) is an antichain. If \( i \in H \) then \( i = h_i \) for some \( i_0 < k \). But then \( \hat{p}_n(i) \geq x_{ih} \), \( \hat{p}_n(i) \geq x_{ij} \), and \( x_{ij} \) and \( x_{ih} \) are incomparable, so since \( T \) is tree-like \( \hat{p}_n(i) \) and \( \hat{p}_n(i) \) are incomparable.

Step 7: Suppose that \( C \subseteq B \) is uncountable and \( U(C) \) holds. Then there is no set \( D \subseteq C \) such that card \( D > n^2 \) and \( D^i \) is an antichain for each \( i < n \).

Proof: Suppose otherwise. Since \( U(C) \) holds and since \( \{y: y \leq x\} \) is countable for all \( x \in T \) the set \( E = \{p \in C: \hat{p}(i) \leq \hat{q}(j) \text{ for some } i,j < n \text{ and } q \in D\} \) is countable. Let \( p \in C - E \) and let \( q \in D \). Since \( q \) must be incompatible with \( p \), it follows that there is a pair \( (i,j) \) such that \( \hat{q}(i) < \hat{p}(j) \). But card \( D > n^2 \), so by the familiar pigeonhole principle there exist \( q_1, q_2 \in D \) and a pair \( (i,j) \) with \( q_1 \neq q_2 \) and \( \hat{q}_1(i), \hat{q}_2(i) < \hat{p}(j) \). But then \( \hat{q}_1(i) \) and \( \hat{q}_2(i) \) are comparable since \( T \) is tree-like, and this is a contradiction.

Steps 3, 6, and 7 are contradictory, and therefore * holds for \( n \). This completes the induction and proves Theorem 4.

The following problems are open:

1. Is it consistent with ZFC to assume that for any partial order \( P \), card \( P = \aleph_1 \) and every uncountable subset of \( P \) contains an uncountable antichain, then \( P \) can be embedded in the rationals?

2. (Galvin, unpublished) Is it consistent with ZFC to assume that for any partial order \( P \), if card \( P = \aleph_2 \) and if every \( P' \subseteq P \) with card \( P' \subseteq \aleph_1 \), can be embedded in the rationals, then \( P \) can be embedded in the rationals?

3. (Baumgartner) Is it consistent with ZFC + \( 2^{\aleph_0} = \aleph_1 \) to assume that every Aronszajn tree is embeddable in the rationals?

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