Stability of Plane Poiseuille Flow to Periodic Disturbances of Finite Amplitude, II

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Communicated November 3, 1970

ABSTRACT The stability of plane Poiseuille flow to periodic disturbances of finite amplitude was investigated by expanding each harmonic of the solution in terms of the Orr-Sommerfeld eigenfunctions with coefficients which are functions of time. The system of nonlinear ordinary differential equations for the coefficients was solved, and the number of harmonics \( N \) was extended from 3, of the previous investigation, to 5. The shift in the neutral curve in going from \( N = 3 \) to \( N = 5 \) is considerable, indicating insufficient convergence. The higher-order harmonics are effective because the zone of mode-coalescence rises with increasing \( N \).

It is known [1] that plane Poiseuille flow

\[
    u_0 = \frac{\partial \psi}{\partial y} = 1 - y^2, \quad v_0 = -\frac{\partial \psi}{\partial x} = 0, \quad (1)
\]

becomes unstable to infinitesimally small disturbances when the Reynolds number \( R \) exceeds the value \( R_c = 5772 \). In a previous investigation [2] we have studied the effect of finite amplitude disturbances, which are restricted to be periodic in \( x \). The method followed was to solve the Navier-Stokes equation

\[
\frac{\partial \nabla \psi}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \nabla \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \nabla \psi}{\partial y} = \frac{1}{R} \nabla \psi, \quad (2)
\]

with

\[
\psi = \varphi_0 + \sum_{n=-\infty}^{\infty} f_n(y,t)e^{-i\alpha_n y}, \quad f_{-n} = \overline{f_n}, \quad (3)
\]

the bar denoting complex conjugate. Substitution of (3) in (2) leads to the system of equations for the \( f_n(y) \)

\[
f_n^{IV} - 2\alpha^2nf_n + \alpha^4nf_n - \frac{\partial R}{\partial t} (\overline{f_n} - \overline{f_n})
\]

\[+ i\alpha R(1 - y^2 + \overline{f_n}(f_n - \overline{f_n}) + (2 - \overline{f_n})f_n) = i\alpha RK_n, \quad (4)
\]

\[
K_n = \sum_{k=1}^{n-1} [kf_{n-k} - (n - k)f_{n-k}]
\]

\[+ \sum_{k=1}^{n} [-kf_{n+k} - (n + k)f_{n+k} + kf_{n+k}] + (n + k)f_{n+k}, \quad (n \geq 1), \quad (5)
\]

where

\[
g_n = f_n - \alpha^2nf_n. \quad (6)
\]

The boundary conditions on the \( f_n \) are

\[
f_n = \overline{f_n} = 0, \quad n > 0, \quad y = \pm 1. \quad (7)
\]

The mean flow is given by

\[
\bar{u}(y,t) = \frac{\partial \psi}{\partial y} = 1 - y^2 + \bar{f}_0(y,t). \quad (8)
\]

On the assumption that the mean pressure-gradient remains unchanged by the perturbation, the equation for \( \bar{f}_0 \) integrates into

\[
\bar{f}_0 - R \frac{\partial \bar{f}_0}{\partial t} = 2\alpha Rsm \sum_{k=1}^{n} kf_k f_k. \quad (9)
\]

The boundary conditions on \( \bar{f}_0 \) are then

\[
\bar{f}_0 = \overline{\bar{f}_0} = 0, \quad y = \pm 1. \quad (10)
\]

![Fig. 1](image)

**FIG. 1.** The critical value of the amplitude \( \lambda_c \) as a function of the Reynolds number \( R \), for the case of a disturbance represented by the first mode of the Orr-Sommerfeld equation. •, unstable; O, stable. \( N \) denotes the number of harmonics included.

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The condition of the vanishing of $f_0$ at the walls stems from the assumed invariance of the mean pressure-gradient.

A suitable representation for $f_0$ is

$$f_0(y, t) = \sum_{\sigma = 1}^{s} A_{\sigma}(t) \cos[(\sigma - \frac{1}{2}) \pi y],$$  \hspace{1cm} (11)$$

while for $n > 0$ we put

$$f_n(y, t) = \sum_{k = 1}^{N} B_{k}^{n}(t) \phi_{k}^{n}(y),$$  \hspace{1cm} (12)

where $\phi_{k}^{n}(y)$ are the solutions of the Orr-Sommerfeld equation

$$\dot{\phi}_{k}^{n} - \alpha^2 n^2 \phi_{k}^{n} + i \alpha n \gamma \phi_{k}^{n} + 2 \phi_{k}^{n} = 0,$$  \hspace{1cm} (13)$$

$$\phi_{k}^{n} = \bar{\phi}_{k}^{n} - \alpha^2 n^2 \phi_{k}^{n}.$$  \hspace{1cm} (14)

Substituting (11) and (12) into (4) and (5), and using the orthogonality condition

$$\int_{-1}^{1} g_{k} \bar{\phi}_{k}^{n} = \delta_{nk},$$  \hspace{1cm} (15)$$

where $\bar{\phi}_{k}^{n}$ is the adjoint solution to (13), we are led to a system of nonlinear ordinary differential equations for the functions $A_{\sigma}(t)$ and $B_{k}^{n}(t)$:

$$\frac{dA_{\sigma}}{dt} = -\pi^2(\sigma - \frac{1}{2})^2 \times$$

$$A_{\sigma} - 4 \alpha \delta m \sum_{n = 1}^{N} \sum_{k = 1}^{N} \sum_{i = 1}^{N} \bar{B}_{k}^{n} B_{i}^{n} \Gamma_{kk},$$  \hspace{1cm} (16)$$

$$\frac{dB_{k}^{n}}{dt} = i \alpha \gamma B_{k}^{n} + 2 i \alpha \sum_{\sigma = 1}^{s} \sum_{i = 1}^{N} A_{\sigma} B_{i}^{\sigma} \Gamma_{kk}^{n} +$$

$$2 i \alpha \sum_{i = 1}^{N} \sum_{j = 1}^{N} \sum_{l = 1}^{N} \bar{B}_{i}^{1} B_{j}^{\sigma - 1} V_{kl},$$  \hspace{1cm} (17)$$

The coefficients in Eqs. (16) and (17) are constants which can be predetermined from the Orr-Sommerfeld eigenfunctions:

$$\Gamma_{kk}^{n} = \int_{0}^{1} \cos[\pi(\sigma - \frac{1}{2}) y] \phi_{k}^{n}(y) \phi_{k}^{n}(y) dy,$$  \hspace{1cm} (18)$$

$$Q_{kk}^{n} = \int_{0}^{1} \phi_{k}^{n} \cos[\pi(\sigma - \frac{1}{2}) y] g_{k}^{n} +$$

$$\pi^2(\sigma - \frac{1}{2}) \phi_{k}^{n} dy,$$  \hspace{1cm} (19)$$

$$V_{kk}^{n} = \int_{0}^{1} \phi_{k}^{n} [(n - l) \phi_{l}^{n} \phi_{j}^{n} - l \phi_{l}^{n} \phi_{j}^{n}] dy,$$  \hspace{1cm} (20)$$

$$U_{kk}^{n} = \int_{0}^{1} \phi_{k}^{n} [(n + l) \phi_{l}^{n} \phi_{j}^{n} - l \phi_{l}^{n} \phi_{j}^{n}] dy.$$  \hspace{1cm} (21)

**DISCUSSION OF RESULTS**

We have solved Eqs. (16) and (17), going up to $N = 5$ in the number of harmonics in (3). In order to assure sufficient convergence, we had to take $s = 60$ terms in (11). For the
number of modes \( k \) in each harmonic of (12) we limited ourselves to \( k = 21 \). This limitation proved to be serious because the zone of coalescence [2] of the interior-modes \((c_r \sim 1)\) with the boundary-modes \((c_r \sim 0)\) rises with increasing order of the mode. Thus, in the case \( R = 2000 \) shown in Fig. 2, the zone of coalescence for the fifth mode \((\alpha = 5)\) has risen to \( k = 16, 17 \). The method of expansion in Orr-Sommerfeld eigenfunctions is therefore nonuniform in that the higher the harmonic, the greater the number of modes required to cover the sensitive zone of coalescence. The effect of the higher harmonics is manifested in the considerable change in the neutral curve which occurs in going from \( N = 3 \) to \( N = 5 \), as shown in Fig. 1.

The curve for \( N = 3 \) in Fig. 1 is different from that shown in Fig. 11 of ref. 1, because in the previous investigation [2] we did not meet the symmetry requirement [3, 4] that the \( f_k(y) \) of even order in (4) are odd functions of \( y \).

That the parity (with respect to \( y \)) of the solution is changed due to the effect of the quadratic terms can be seen by an inspection of Eq. (2). If at time \( t = 0 \) the linear terms in (2) are even (odd), then the quadratic terms are odd (even), and their contribution spoils the original parity. Fig. 3 shows the eigenvalues \( c_{21} \) in the complex \( c \)-plane for the odd eigenfunction \( \phi_c \) of the second mode. As in the case of even eigenfunction of the first mode shown in Fig. 4 of ref. 1, the low order modes separate into the interior- and boundary-types, which merge at \( k = 9, 10 \). The growth of the disturbance when the initial amplitude \( \lambda \) exceeds the critical value \( \lambda_0 \) is shown in Fig. 4, and is similar in appearance to the pattern of the curve shown in Fig. 3 of ref. 1. One finds similarly that for \( R = 5000 \), where a \( B_{11} \) type of disturbance \( \lambda_0 = 0.00055 \) at \( N = 5 \), in the case of a \( B_{11} \) type of initial disturbance \( \lambda_1 \), is reduced to 0.00015, \( B_{11} \) being situated in the sensitive zone of coalescence (Fig. 4 of ref. 1).

The method of expansion in Orr-Sommerfeld eigenfunctions looked promising at first because it is a good approximation in the early stages of the disturbance and it affords an exact satisfaction of the boundary conditions, while making it possible to reduce the solution of the Navier-Stokes equation to the more tractable task of solving a system of nonlinear ordinary differential equations. A detailed mapping of the Orr-Sommerfeld eigenvalues then revealed [2] the existence of the two distinct types of modes, interior and boundary, and their subsequent merging. Since, however, the zone of coalescence was found to occur at higher order modes as the order of the harmonic is increased, the convergence of the expansion turned out to be slow. It is therefore premature to decide on the physical importance of the phenomenon of mode-coalescence, before the possibility is excluded that the contributions from the zones of coalescence of the various modes may tend to cancel themselves out. It would be of interest to study the convergence of the method with the newer types of computers by going to higher order harmonics, and also to attempt a direct finite-difference attack on the Stokes-Navier equation.

We are indebted to Mr. Benjamin Gabai for assistance in programming. This investigation was supported by the National Science Foundation (grant GA-1062).

4. Pekeris, C. L., and B. Shkoller, J. Fluid Mech., 39, 631 (1969). Here, the \( F_i(y) \) are odd functions of \( y \) when \( i \) is even.