Solution of the Time-Dependent Schrödinger Equation by the Laplace Transform Method

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ABSTRACT The time-dependent Schrödinger equation for two quite general types of perturbation has been solved by introducing the Laplace transforms to eliminate the time variable. The resulting time-independent differential equation can then be solved by the perturbation method, the variation method, the variation-perturbation method, and other methods.

Conventionally, the time-dependent Schrödinger equation is solved by the time-dependent perturbation method, which is based on a power series expansion in terms of the strength of the applied perturbation. The solution of the time-dependent Schrödinger equation is the basis of calculating the transition probabilities for various processes, including the interaction of an electromagnetic field with matter, energy relaxation, and energy transfer.

Recently, the time-dependent Schrödinger equation of semi-classical radiation has been solved by Karplus and Kolker (1, 2) by using the so-called variation-perturbation approach. According to this approach, the time-dependent wave function is expanded in terms of a power series of a perturbation parameter by regarding the interaction of radiation with the system as a perturbation, and the time-dependent Schrödinger equation is then separated into an infinite set of coupled equations according to the order of the perturbation parameter. This set of coupled equations is solved by the variation method after suitably eliminating the time variable.

In the present investigation, we are concerned with the solution of the time-dependent Schrödinger equation by using the Laplace transform method (3). It is well known that under certain conditions one can eliminate the independent variable t, time, from partial differential equations by introducing the Laplace transformation (4-6). It will be shown that after introducing the Laplace transform into the time-dependent Schrödinger equation, the resulting time-independent differential equation can be solved by using the variation method (7) or other approximation methods (7).

GENERAL CONSIDERATION

The time-dependent Schrödinger equation is given by

$$ H \Psi = i\hbar \frac{\partial \Psi}{\partial t} $$

where $H = H_0 + H'$, where $H'$ is the perturbation. Introducing the Laplace transform (3)

$$ x(p) = \mathcal{L}[x(t)] = \int_0^\infty e^{-pt} x(t) \, dt $$

into Eq. 1, we obtain

$$ H^0 \Psi(p,r) + \mathcal{L}[H'(r,t)] = -i\hbar \Psi(r,0) + i\hbar p \Psi(p,r). $$

In this paper, we shall consider the case in which $H'$ is time-independent and the case in which $H'$ can be expressed as (8)

$$ H' = \sum_i V_i(r) \exp(-ilt/\hbar) $$

where $l$ represents the running index and may take values from $-\infty$ to $\infty$. Actually the first case where $H'$ is time-independent is just a special case of the second case in which $V_0(r) \neq 0$, and $V_i(r) = 0$, for $l \neq 0$. But since the solutions of the Schrödinger equation to be presented here are different for the two types of $H'$, we shall discuss these two cases separately.

If $H'$ is time-independent, Eq. 3 becomes

$$ (H_0 + H' - i\hbar p) \Psi(p,r) = -i\hbar \Psi(r,0) $$

which is independent of time. Eq. (5) is simply a linear variation of

$$ M = \langle \Psi^+(p,r) | H - i\hbar p | \Psi(p,r) \rangle + i\hbar \langle \Psi^+(p,r) | \Psi^+(r,0) | \Psi(p,r) \rangle $$

with respect to $\Psi(p,r)$. $\Psi^+(p,r)$ in Eq. (6) represents the adjoint of $\Psi(p,r)$. Since $M$ is not real, to retrieve Eq. (3) the extremals of $|M|^2$ are required. In other words, $\Psi(p,r)$ is the variational solution of the stationary values of $|M|^2$ given in Eq. (6).

Notation: $H$, Hamiltonian; $r$, vector $r$; $\hbar$, $\hbar/2\pi$. 76
Next we turn to the second case for which we have
\[ H^0 \Psi(p,r) + \sum_i V_i(r)L[\exp(-iEt/h)\Psi(r,0)] = -i\hbar \Psi(r,0) + i\hbar p \Psi(p,r). \] (7)
By using the relation, Eq. (A-2) given in the Appendix, we obtain
\[ H^0 \Psi(p,r) + \sum_i V_i(r)\exp(iEt\hbar/h)\Psi(p,r) = -i\hbar \Psi(r,0) + i\hbar p \Psi(p,r), \] (8)
or
\[ [H^0 + \sum_i V_i(r)\exp(itEd\hbar/h)] \Psi(p,r) = -i\hbar \Psi(r,0), \] (9)
Comparing Eqs. (5) and (6) with Eq. (9), we can see that Eq. (9) is just a result of a linear variation of
\[ N = \langle \Psi^+(p,r) \mid H^0 + \sum_i V_i \exp(itEd\hbar/h) \rangle - i\hbar p \langle \Psi(p,r) \rangle + i\hbar \langle \Psi^+(p,r) \rangle \Psi(r,0) \rangle + -i\hbar \langle \Psi(r,0) \mid \Psi(p,r) \rangle \] (10)
or equivalently, Eq. (9) can be obtained as the extremal of \( |N|^2 \). Thus in the above discussion, we have shown how to use the Laplace transform to transform the time-dependent Schrödinger equations for two quite general types of perturbations into time-independent differential equations that can be solved by the variational method. Of course, Eqs. (9) and (5) can also be solved by using the perturbation method or other methods like the Green function method, etc. (7).

Once the Laplace transform of the time-dependent wave function is determined from Eq. (5) or Eq. (9), the time-dependent wave function \( \Psi(r,t) \) can be obtained by the inverse Laplace transform (3) as follows
\[ \Psi(r,t) = \frac{1}{2\pi i} \int \gamma \exp\{i\hbar H\Psi(p,t)dp \} \] (11)
where the contour \( \gamma \) represents the path of integration taken along the imaginary axis shifted by an arbitrarily small positive quantity and the semicircle of infinite radius on the left half of the complex \( p \)-plane.

A Perturbation That is Time-Independent

In this section, we shall discuss the solution of the time-independent Schrödinger equation of a time-independent perturbation. In this case, we have to solve Eq. (5). To solve Eq. (5), firstly, we use the perturbation method. For this purpose, we rewrite Eq. (5) as
\[ (H^0 + \lambda H') \Psi(p,r) = -i\hbar \Psi(r,0) \] (12)
where \( \lambda \) is the perturbation parameter. Expanding \( \Psi(p,r) \) in \( \lambda^k \),
\[ \Psi(p,r) = \sum_{n=0}^{\infty} \lambda^n \Psi^{(n)}(p,r) \] (13)
substituting Eq. (12) into Eq. (13) and separating the terms of each order in \( \lambda \), we obtain a set of coupled equations as,
\[ (H^0 - i\hbar) \Psi^{(0)}(p,r) = -i\hbar \Psi(r,0) \] (14-A)
\[ (H^0 - i\hbar) \Psi^{(1)}(p,r) + H^0 \Psi^{(0)}(p,r) = 0 \] (14-B)
\[ (H^0 - i\hbar) \Psi^{(2)}(p,r) + H^0 \Psi^{(1)}(p,r) = 0 \] etc. (14-C)
For convenience, we suppose that the system is in the state \( \varphi_0(p) \) at \( t = 0 \), i.e., \( \Psi(r,0) = \varphi_0(p) \). To solve the coupled equations of Eq. (14), as usual we expand \( \Psi(n)(p,r) \) in terms of the eigen functions of \( H^0 \),
\[ \Psi^{(n)}(p,r) = \sum_{m} C_{nm}(p) \varphi_m(p) \] (15)
Substitution of Eq. (15) into Eq. (14-A) yields
\[ C_{n1}(p) = \frac{i\hbar}{i\hbar - E_1}\varphi_0(p) \] (16)
Substitution of Eqs. (15) and (16) into Eq. (14-B) yields
\[ C_{nm}(p) = \frac{i\hbar H_{m1}'}{(i\hbar p - E_m)(i\hbar p - E_1)} \] (17)
where \( H_{m1}' = \langle \varphi_0 \mid H^0 \mid \varphi_0 \rangle \).

Thus to the first order approximation, we have
\[ \Psi(p,r) = \frac{i\hbar \varphi_0(p)}{i\hbar p - E_1} + \frac{i\hbar H_{m1}' \varphi_0(p)}{(i\hbar p - E_1)(i\hbar p - E_m)} + \ldots \] (18)
By inverting the Laplace transform of Eq. (18), we obtain
\[ \Psi(r,t) = \left[ 1 - i\lambda / \hbar H_{m1}' \right] \varphi_0(p) \exp(-iEt\hbar / \hbar) + \lambda \sum_{m} \frac{H_{m1}'}{E_m - E_1} \{ \exp(-iEt\hbar / \hbar) - \exp(-iEt\hbar / \hbar) \} \varphi_m \] (19)
or
\[ \Psi(r,t) = \left[ 1 - i\lambda / \hbar H_{m1}' \right] \varphi_0(p) \exp(-iEt\hbar / \hbar) + \sum_{m} \frac{H_{m1}'}{E_m - E_1} \{ 1 - \exp\left(-i(E_1 - E_m) / \hbar \right) \} \exp(-iEt\hbar / \hbar) \varphi_m (r) + \ldots \] (20)
This is what is to be expected (10). It should be noticed that the set of equations, Eq. (14), can be solved by the variation method (2).

Next we solve Eq. (5) by expanding \( \Psi(p,r) \) as
\[ \Psi(p,r) = \sum_{n=1}^{N} C_n(p) \varphi_n(r) \] (21)
If the expansion in Eq. (21) covers a complete set of \( \varphi_n(r) \), the \( \Psi(p,r) \) determined and hence \( \Psi(r,t) \) will be exact. Otherwise if the expansion in Eq. (21) does not
cover a complete set of \( \varphi_n(r) \), or \( \varphi_n(r) \) used are only approximate, then from the discussion of the previous section \( \Psi(p,r) \) determined and hence \( \Psi(r,t) \) will only represent an approximate solution of the Schrödinger equation. Substituting Eq. (21) into Eq. (5), we have

\[
\sum_{n} C_{n}(p) (H^{p} + H' - i\hbar)p_{n}(r) = -i\hbar\Psi(r,0). \tag{22}
\]

If \( \varphi_n(r) \)'s are orthonormal, and \( \Psi(r,0) = \varphi_0(r) \), multiplying both sides of Eq. (22) by \( \varphi_n^*(r) \) and integrating over coordinates, we obtain

\[
(i\hbar p - E_n)C_n(p) - \sum_{n} C_{n}(p) H_{nn}' = i\hbar \delta_{n1} \tag{23}
\]

where \( \delta_{n1} \) represents the Kronecker delta and \( E_n = \langle \varphi_n | H^{p} | \varphi_n \rangle \). Eq. (23) indicates that to determine \( C_{n}(p) \), we have to solve the set of simultaneous equations given in Eq. (23). For this purpose we let \( \Delta_n(i\hbar p) \) represent the \( N \times N \) determinant obtained from the coefficients of \( C_n(p) \).

\[
\Delta_n(i\hbar p) = \begin{vmatrix}
(i\hbar p - E_1 - H_{11}'), & -H_{12}', \\
-H_{11}', & (i\hbar p - E_2 - H_{22}'), \\
-H_{12}', & -H_{22}' \\
\vdots & \vdots \\
-H_{N1}', & -H_{N2}', \\
\end{vmatrix} \tag{24}
\]

From Eqs. (23) and (24), \( C_n(p) \) can be solved as (11)

\[
C_n(p) = -i\hbar \frac{\partial}{\partial H_{11}'} (\log \Delta_n) \quad (n = 1,2,3 \ldots N) \tag{25}
\]

where \( H_{11}' \) are assumed to be independent of \( H_{11}' \).

Since \( \Delta_n(i\hbar p) \) represents an \( N \)-th power polynomial in \( i\hbar p \), \( \Delta_n(i\hbar p) \) can be written as

\[
\Delta_n(i\hbar p) = \prod_{j=1}^{N} (i\hbar p - \lambda_j). \tag{26}
\]

Introducing Eq. (26) into Eq. (25), we have

\[
C_n(p) = i\hbar \sum_{j=1}^{N} \frac{\partial \lambda_j / \partial H_{11}'}{i\hbar p - \lambda_j} \quad (n = 1,2,3 \ldots N). \tag{27}
\]

Substituting Eq. (27) into Eq. (21) and applying the inverse transform to the resulting expression, we obtain

\[
C_n(t) = \sum_{j=1}^{N} \frac{\partial \lambda_j}{\partial H_{11}'} \exp(-i\lambda_j t/\hbar) = \frac{\partial}{\partial H_{11}'} \left( \sum_{j=1}^{N} \exp(-i\lambda_j t/\hbar) \right) \tag{28}
\]

and

\[
\Psi(r,t) = \sum_{n=1}^{N} \sum_{j=1}^{N} \varphi_n(r) \frac{\partial \lambda_j}{\partial H_{11}'} \exp(-i\lambda_j t/\hbar) = \sum_{n=1}^{N} \varphi_n(r) \frac{\partial}{\partial H_{11}'} \left[ \sum_{j=1}^{N} \exp(-i\lambda_j t/\hbar) \right]. \tag{29}
\]

As an application of Eq. (28), we consider a two-state system (12). For this system, we have

\[
\Delta_2(i\hbar p) = (i\hbar p - \lambda_1)(i\hbar p - \lambda_2) \quad \tag{30}
\]

where

\[
\lambda_1 = \frac{1}{2} (\epsilon_1 + \epsilon_2 + D_{12}) \quad \text{and} \quad \lambda_2 = \frac{1}{2} (\epsilon_1 + \epsilon_2 - D_{12}) \quad \tag{31}
\]

\( \epsilon_1, \epsilon_2 \) and \( D_{12} \) are defined as

\[
\epsilon_1 = E_1 + H_{11}'; \quad \epsilon_2 = E_2 + H_{22}'; \quad \text{and} \quad D_{12} = [(\epsilon_2 - \epsilon_1)^2 + 4H_{12}'H_{22}']^{1/2}. \tag{32}
\]

Thus from Eq. (29), we obtain \( C_1(t) \) and \( C_2(t) \) as

\[
C_1(t) = \exp(-it(\epsilon_1 + \epsilon_2)/\hbar) \left[ \cos \frac{D_{12} t}{2\hbar} + \frac{i(\epsilon_2 - \epsilon_1)}{D_{12}} \sin \frac{D_{12} t}{2\hbar} \right] \tag{33}
\]

and

\[
C_2(t) = -2i \frac{H_{12}'}{D_{12}} \exp(-it(\epsilon_1 + \epsilon_2)/\hbar) \sin \frac{D_{12} t}{2\hbar}. \tag{34}
\]

It can easily be shown that

\[
|C_1(t)|^2 + |C_2(t)|^2 = 1. \tag{35}
\]

So far the solution is exact. Now suppose we set \( C_1(t) = \exp(-it\Gamma t/\hbar - \Gamma t/2) \), where \( \Gamma = \Gamma_r + i\Gamma_i \). Comparing this expression with that given in Eq. (33), we have

\[
e^{-\Gamma t/2} \cos \left( \frac{E_{12} t}{\hbar} + \frac{\Gamma t}{2} \right) = \cos \frac{D_{12} t}{2\hbar} \cos \left( \frac{\epsilon_1 + \epsilon_2}{2\hbar} + \frac{\Gamma t}{2} \right) \times \sin \frac{D_{12} t}{2\hbar} \sin \left( \frac{\epsilon_1 + \epsilon_2}{2\hbar} \right) \tag{36}
\]
and
\[ e^{-\frac{r^2}{2}} \sin \left( \frac{E_1}{\hbar} \right) = \cos \frac{D_{st} t}{2\hbar} \sin \frac{(\epsilon_2 - \epsilon_1)}{2\hbar} - \frac{(\epsilon_2 - \epsilon_1)}{D_{st}^2} \times \sin \frac{D_{st} t}{2\hbar} \cos \frac{(\epsilon_1 + \epsilon_2)}{2\hbar}. \] (37)

Elimination of \( \Gamma_\ell \) from Eqs. (36) and (37) gives
\[ e^{-\frac{r^2}{2}} = \cos^2 \frac{D_{st}^2}{2\hbar} + \frac{(\epsilon_2 - \epsilon_1)^2}{D_{st}^2} \sin^2 \frac{D_{st} t}{2\hbar} = 1 - \frac{4H_{ll}'H_{ll}'}{D_{st}^2} \sin^2 \frac{D_{st} t}{2\hbar}. \] (38)

If \( t\ell' \ll 1 \) but \( t \gg \hbar/D_{ll} \), then (10)
\[ \Gamma_\ell = \frac{2\pi}{\hbar} H_{ll}'H_{ll}' \left( \frac{D_{st}}{D_{ll}} \right) = \frac{2\pi}{\hbar} H_{ll}'H_{ll}' \delta(D_{st} \hbar) \] (39)

which, of course, is the transition probability of the transition \( 1 \rightarrow 2 \). If \( (\epsilon_2 - \epsilon_1)^2 \gg 4H_{ll}'H_{ll}' \), then
\[ \Gamma_\ell = \frac{2\pi}{\hbar} |H_{ll}'|^2 \delta(E_1 - E_1 - \Delta E) \] (40)

where \( \Delta E = H_{ll}' - H_{ll}' + 2|H_{ll}'|^2/(E_{ll} - E_{ll}) \). The conditions of \( t\ell' \ll 1 \) and \( t \gg \hbar/D_{ll} \) indicate that \( \Gamma_\ell \ll D_{ll}/\hbar \) or \( \hbar \Gamma_\ell \ll D_{ll} \), which means that the line breadth is small, a condition well satisfied for all atomic and molecular systems. Thus under these conditions the real part of \( \Gamma \) yields the damping factor and the imaginary part of \( \Gamma \) gives rise to the level shift (10).

We shall discuss another method to solve the set of equations given in Eq. (23). As with Heitler and Ma (13), we set, for \( n \neq 1 \)
\[ C_n(p) = U_n(p)C_1(p) \zeta(\hbar p - E_n) \] (41)

where \( U_n(p) \)'s are to be determined and \( \zeta(\hbar p - E_n) \) represents the \( \zeta \)-function first introduced by Dirac (14). In many cases when \( \hbar p \) is not equal to \( E_n \), one can simply replace \( \zeta(\hbar p - E_n) \) in Eq. (41) by
\[ \frac{1}{\hbar p - E_n}. \] But to be general, we retain \( \zeta(\hbar p - E_n) \) in the discussion. Substituting Eq. (41) into Eq. (23), we have, for \( n \neq 1 \)
\[ U_n(p) = H_{ll}' + \sum_{m=1}^{N} U_m(p)H_{ll}' \zeta(\hbar p - E_m) \] (42)

and for \( n = 1 \)
\[ (\hbar p - E_1) C_1(p) - H_{ll}'C_1(p) - C_1(p) \sum_{m=1}^{N} U_m(p)H_{ll}' \zeta(\hbar p - E_m) = \hbar. \] (43)

Here for convenience, we set \( U_1(p) = 0 \). Thus from Eq. (42), we can calculate \( U_n(p) \) \( (n = 2, 3, \ldots N) \) and after determining \( U_n(p) \), we can solve for \( C_1(p) \) from Eq. (43) and then \( C_n(p) \) from Eq. (41). If we define
\[ \frac{\hbar}{2} \Gamma_1(p) = -H_{ll}' + \sum_{m=1}^{N} U_m(p)H_{ll}' \zeta(\hbar p - E_m) \] (44)

then
\[ C_1(p) = \frac{\hbar}{\hbar p - E_1 + \frac{\hbar}{2} \Gamma_1(p)} \] (45)

and
\[ C_n(p) = \frac{\hbar U_n(p) \zeta(\hbar p - E_n)}{\hbar p - E_1 + \frac{\hbar}{2} \Gamma_1(p)} \] (46)

By inverting the Laplace transform of Eqs. (45) and (46), we obtain
\[ C_1(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dp e^{pt}}{p + iE_1/\hbar + \frac{1}{2} \Gamma_1(p)} \] (47)

and
\[ C_n(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{U_n(p) \zeta(\hbar p - E_n) e^{pt} dp}{p + iE_1/\hbar + \frac{1}{2} \Gamma_1(p)} \] (48)

\( \Gamma_1(p) \) defined by Eq. (44) is, of course, related to the damping factor. It can readily be shown from Eq. (44) that the real part of \( \Gamma_1(p) \) corresponds to the sum of all transition probabilities originating from \( \varphi_1(r) \), and the imaginary part of \( \Gamma_1(p) \) corresponds to the level displacements, under the conditions of \( t\ell \ll 1 \) and \( \hbar \Gamma_1(p) \ll |E_n - E_1| \). The damping phenomenon has been much discussed by Heitler and Ma, and others (15-18) and will not be discussed here.

To demonstrate the application of Eqs. (41-46), we again consider the two-state systems discussed above. For this system, we obtain from Eq. (42)
\[ U_2(p) = \frac{(\hbar p - E_1) H_{ll}'}{\hbar p - E_1 - H_{ll}'} \] (49)

and from Eqs. (45) and (46)
\[ C_1(p) = \frac{\hbar (\hbar p - E_1)}{(\hbar p - E_1)(\hbar p - E_1 - H_{ll}')}, \] (50)

\[ C_2(p) = \frac{\hbar H_{ll}'}{(\hbar p - E_1)(\hbar p - E_1 - H_{ll}')}. \] (51)

By inverse Laplace transform of \( C_1(p) \) and \( C_2(p) \) given in Eqs. (50) and (51), we obtain the exact solution for
C_n(t) and C_t(t) which are given in Eqs. (33) and (34), respectively. Based on the discussion of the two-state system above, Eqs. (49–51), we can develop an iteration solution of Eqs. (42) and (43), which will give a good approximate solution for many-state systems but will give an exact solution for two-state systems. As a first iteration approximation, from Eq. (42), we get

\[ U_n(p) = H_{n1}' + U_n(p)H_{n2}'(\text{i}\hbar p - E_n) \]  

(52)

or

\[ U_n(p) = \frac{H_{n1}'}{1 - H_{n2}''(\text{i}\hbar p - E_n)} . \]  

(53)

Substitution of Eq. (53) into Eq. (44) gives

\[ \frac{i\hbar}{2} \Gamma_n(p) = -H_{n1}' - \sum_{m \neq n} H_{1m}'H_{m2}'(\text{i}\hbar p - E_m) \]  

(54)

Combining Eq. (54) with Eqs. (55) and (56), we obtain the first iteration solution of C_l(p) and C_u(p),

\[ C_l(p) = \frac{i\hbar}{H_{n1}'}(\text{i}\hbar p - E_n) \]  

(55)

\[ C_u(p) = \frac{H_{n1}'}{1 - H_{n2}''(\text{i}\hbar p - E_n)} \]  

(56)

This process can be repeated to obtain the higher order approximations. Next it will be shown that the first iteration solution of Eqs. (42) and (43) just described simply corresponds to expanding the determinant \( \Delta_N'(\text{i}\hbar p) \) as follows (11),

\[ \Delta_N'(\text{i}\hbar p) = \prod_{n=1}^{N} (\text{i}\hbar p - E_n - H_{n1}') - \sum_{m>n}^{N} H_{m1}'H_{m2}'(\text{i}\hbar p - E_m) - \sum_{m>n}^{N} H_{m1}'H_{m2}'(\text{i}\hbar p - E_m) - \cdots \]  

(57)

which results from using Cayley’s theorem for the expansion of a determinant (11). Up to this approximation of \( \Delta_N'(\text{i}\hbar p) \) we can readily see that \( C_l(p) \) and \( C_u(p) \) given in Eqs. (55) and (56) can be reproduced by substituting Eq. (57) into Eq. (25) or Eq. (27). The higher order approximations by the use of Cayley’s theorem for the expansion of a determinant corresponds to the higher order iteration solutions. The application of Cayley’s theorem in solving the secular determinant has been discussed recently by Sasakawa (19) and Chan (20).

The above approach can be applied to the discussion of radiative and nonradiative processes, and energy-transfer in molecular luminescence (21) and ORD, MOR, CD, MCD, etc. (22, 23). The time-dependent Schrödinger equation for the perturbation that has the time dependence shown in Eq. (4) can be solved similarly (compare Eqs. 12–57) and will not be produced here (24).

In concluding the discussion, it should be noted that as usual the difference between the perturbation and variation methods in solving the differential equation resulting from the Laplace transforms of the time-dependent Schrödinger equation, and hence in calculating the physical properties resulting from it, lies in the fact that the former usually gives rise to an infinite series which is difficult to handle, while the latter does not have this difficulty. But the accuracy of the calculation using the variation method depends on the trial wave function used. Also, the damping effect can be taken into account easily by the variation method, when the damping effect is important. Although in this investigation, we use the Laplace transform method to eliminate the time variable from the time-dependent Schrödinger equation, the same purpose can also be accomplished by using the Fourier transforms.

APPENDIX

From the definition of the Laplace transform

\[ \hat{x}(p) = L[x(t)] = \int_{0}^{\infty} e^{-pt} x(t) \, dt \]  

(A-1)

we obtain

\[ \hat{x}(p + q) = \int_{0}^{\infty} e^{-(p+q)t} x(t) \, dt = \exp(qD_p) \hat{x}(p) \]  

(A-2)

by Taylor’s expansion, where \( D_p \) represents the differential operator, \( D_p = \frac{\partial}{\partial p} \). The following relations can be easily derived for the operator \( \exp(qD_p) \):

\[ \exp(qD_p)\exp(qD_p) = \exp(q_1 + q)D_p = \exp(qD_p)\exp(qD_p) \]  

(A-3)

\[ \exp(qD_p)\exp(-qD_p) = 1 \]  

(A-4)

and

\[ [1 + a \exp(qD_p)]^{-1} = \sum_{n=0}^{\infty} (-1)^n a^n \exp(nqD_p) \]  

(A-5)

where \( a \) does not depend on \( p \). Also, we have

\[ \exp(qD_p)[u(p) v(p)] = \sum_{n=0}^{\infty} \frac{q^n}{n!} D_p^n[u(p) v(p)] = \]  

\[ \sum_{n=0}^{\infty} \frac{q^n}{n!} D_p^n \sum_{m=0}^{n} \frac{q^m}{m!} D_p^m \]  

(A-6)
The above relation can be easily generalized.
\[
\exp(qDv)[u_1(p) u_2(p) \ldots u_n(p)] = \prod_{i=1}^{n} \exp(qDv_i) u_i(p)
\]
\[
[u_1(p) \ldots [\exp(qDv_i) u_i(p)] = u_i(p + q) u_i(p + q) \ldots u_i(p + q).
\]

(A-7)

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