Stochastic Speculative Price

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ABSTRACT Because a commodity like wheat can be carried forward from one period to the next, speculative arbitrage serves to link its prices at different points of time. Since, however, the size of the harvest depends on complicated probability processes impossible to forecast with certainty, the minimal model for understanding market behavior must involve stochastic processes. The present study, on the basis of the axiom that it is the expected rather than the known-for-certain prices which enter into all arbitrage relations and carryover decisions, determines the behavior of price as the solution to a stochastic-dynamic-programming problem. The resulting stationary time series possesses an ergodic state and normative properties like those often observed for real-world bourses.

CERTAINTY RELATIONS

The simplest formal theory of speculative price [1] assumes that demands of different periods are independent functions relating consumption, \( c_t \), to \( p_t \), \( p_t = P[c_t] \), \( P' < 0 \), and that costs of carryover consist only of interest and "shrinkage" charges. The equation of price then becomes

\[
p_t = P[H_t + aqh_t - q_t], \quad (t = 0, 1, 2, \ldots) \tag{1}
\]

where \( a \) is the fraction of grain carried over at a time, \( t - 1 \), which is left after shrinkage to be available at time \( t \), \( q_t \) is the amount of grain carried over at time \( t \) for availability at time \( t + 1 \), \( r \) is the interest rate, and \( H_t \) is the harvest at time \( t \). When \( H_t \) is foreseeable with certainty, competitive arbitrage would enforce the intertemporal-equilibrium conditions

\[
(1 + r)^{-1}aP[H_t + aq_t - q_{t+1}] - P[H_t + aq_{t-1} - q_t] \leq 0
\]

\[
q_t(1 + r)^{-1}aP[H_t + aq_t - q_{t+1}] - P'[H_t + aq_{t-1} - q_t] - 0, \quad (t = 0, 1, \ldots, T)
\]

\[
[q_t, q_T] \text{ specified.} \tag{2}
\]

These second-order dynamic difference equations and inequalities can be solved for the unknowns \([q_0, q_1, \ldots, q_{T-1}; c_0, c_1, \ldots, c_T; p_0, \ldots, p_T]\). It can be shown that these unknowns are obtainable as the maximizing solution to a variational problem of optimal-control type, namely

\[
\max_{q_{t-1}} \sum_{a_{t-1}} (1 + r)^{-1}U[H_t + aq_{t-1} - q_t] \tag{3}
\]

where the \([H_0, \ldots, H_T; q_0, q_T]\) are prescribed and all the decision variables are restricted to being nonnegative, \( q_t \geq 0 \), and where

\[
U[c] = \int P[x]dx, P[c] = U'[c], U' < 0 \tag{4}
\]

Applying the Kuhn-Tucker conditions to the solution of (3) is seen to yield the market behavior conditions of (2).

PROBABILITY MODEL

It is assumed that the harvests \([H_t]\) are not foreseeable with certainty but can be taken as subject to known probability distributions that are uniform through time and independent, namely

\[
\text{Prob} \{H_t \leq h\} = F(h), \quad \text{Prob} \{H_t \leq h_0 \text{ and } H_{t+1} \leq h_1\} = F(h_0)F(h_1) \tag{5}
\]

Since exact arbitrage relations like (2) are no longer valid, some new hypothesis is needed. The present model postulates that the mathematical expectation of price plays the role in the decisions of the stochastic model about carryover that known-future-price plays in the deterministic model.

Postulate: Positive carryover will take place only if interest and shrinkage costs are just covered by expected future price:

\[
(1 + r)^{-1}aE_p^T - p_t \leq 0
\]

\[
q_t(1 + r)^{-1}aE_p^T - p_t = 0 \tag{6}
\]

where \( E \) stands for the expected value operator: e.g.,

\[
EH_t = \int_0^\infty hdF(h)
\]

a Stieltjes integral that handles both discrete probabilities and densities.

Just as (2) was obtainable as the optimizing solution to (3), the conditions of (4) are deducible as the optimizing solution to the following stochastic-dynamic-programming problem: for prescribed \([q_0, q_T]\)

\[
J_T[H_0 + aq_0] = \max_{q_0, \ldots, q_T} \sum_{t=0}^T (1 + r)^{-1}U[H_t + aq_{t-1} - q_t], \quad q_t \geq 0 \tag{7}
\]

The usual Bellman [2] technique solves such a programming problem by recursive sequence

\[
J_n[H_{T-n} + aq_{T-n-1}] = \max_{q_{T-n}} \{ U[H_{T-n} + aq_{T-n-1} - q_{T-n}] + (1 + r)^{-1}EJ_{n-1}[H_{T-n+1} + aq_{T-n}], \quad q_{T-n} \geq 0 \}
\]

\[
(n = 1, 2, \ldots, T) \tag{8}
\]

\[
J_0[H_T + aq_{T-1}] = U[H_T + aq_{T-1} - q_T]
\]
Each of these concave programming problems can be solved by Kuhn-Tucker programming for one (or in case of ties, more than one) optimal carryover strategy
\[ q_{r-n}^* = f_n(H_{r-n} + aq_{r-n-1}), \quad 0 \leq f'_n \leq 1 \] (9)
The dependence of these functions on terminal \( q_r \) is not indicated explicitly.
To solve for the \( q^*_r \), we write out the (Kuhn-Tucker) necessary conditions for the maxima
\[ 0 \geq -U'[H_{r-n} + aq_{r-n-1} - q_{r-n}] + (1 + r)^{-a}aEJ_{aq_{r-n}}'[H_{r-n+1} + aq_{r-n}], \]
\[ 0 = q_{r-n}[-U'[H_{r-n} + aq_{r-n-1} - q_{r-n}] + (1 + r)^{-a}aEJ_{aq_{r-n}}'[H_{r-n+1} + aq_{r-n}]] \] (10)
Because the maximands are all concave, these necessary conditions are also sufficient. Their equivalence with the basic postulate of the model, (6), is obvious upon recalling the definition of identity \( P[c] \equiv U'[c]. \) Worth noting also is the envelope condition implied by the maximum
\[ J_n'[H_{r-n} + aq_{r-n-1}] = U'[H_{r-n} + aq_{r-n-1} - f_n(H_{r-n} + aq_{r-n-1})] \] (11)
This completes the solution for fixed horizon \( T \), permitting derivation of all economic magnitudes as random variables, whose laws of behavior depend upon the exogenous random variable \( H_t \) and the form of the various functions involved.
For unlimited horizon, \( T = \infty \), the variables will form a stationary time series.

**Unlimited horizon**

As \( T \), the length of the horizon becomes ever larger, the dependence on terminal \( q_r \) becomes ever weaker. Actually, \( f_n(\cdot) \) approaches a limiting function \( f(\cdot) \), and also \( J_n(\cdot) \) approaches \( J(\cdot) \) as \( n \to \infty \),
\[ \lim_{T \to \infty} q^*_n = \lim_{T \to \infty} f_{T-n}(H_t + aq_{t-1}) = \lim_{n \to \infty} f_n(H_t + aq_{t-1}) = f(H_t + aq_{t-1}) \] (12)
\[ \lim_{T \to \infty} J_{T-n}(H_t + aq_{t-1}) = J[H_t + aq_{t-1}] \]
These presuppose that regularity conditions and well-behaved transversality conditions are satisfied, such as
\[ \lim_{T \to \infty} (1 + r)^{-T}q_r = 0, \quad \lim_{T \to \infty} (1 + r)^{-T}E_p_{r-T} = 0 \]
The limiting functions themselves satisfy time-free functional equations of Bellman type
\[ 0 \geq -U'[x - f(x)] + (1 + r)^{-a}a\int_0^\infty U'[h + af(x)]dF(h) \]
\[ 0 = f(x)[-U'[x - f(x)] + (1 + r)^{-a}a\int_0^\infty U'[h + af(x)]dF(h)] \] (13)
\[ J[x] = U[x - f(x)] + (1 + r)^{-1}\int_0^\infty J'[h + af(x)]dF(h) \]
These are derived by taking \( T \) limits in (10), using the envelope relation (11), and recalling the definition of the expectation operator. In principle, they can be solved for \( f(x) \) and \( J(x) \) by a variety of methods of successive approximation; in the absence of \( U(c) \) and \( F(h) \) taking specially simple analytic forms, one natural method of solution would be to set \( q_r \) at some reasonable level, between \( \int_0^\infty hdF(h) \) and zero, actually solve recursively for \( f_n(x) \), \( J_n[x] \), and then let \( n \) become large.

**Conditional probabilities**

Once the decision rule for carryover, \( f(x) \), becomes known, \( [q_1], [p_1], \) and \( [c] \) become random variables subject to known probability laws. For brevity, denote \( H_t + aq_{t-1} = x_t \), the stock of grain available after the harvest is in. This also is a random variable.

Economic intuition about how an organized market both does and should work suggests certain reasonable properties for the carryover function \( f(x) \).

1. When, because of a run of recent harvests "above normal," the stock of available inventory is "high," one expects the carryover to be high; i.e., \( f(x) \) increases with \( x \).
2. Each increment of inventory, additional to already high inventory, can be expected to be divided between abnormally high consumption today (high \( c_t \) and low \( p_t \)) and incremental carryover for higher future consumptions: i.e., \( c(x) = x - f(x) \) and \( f(x) \) have the properties \( 0 \leq f(x) \leq 1 \), and the price inequalities of (6) will hold.
3. When, because of a run of recent harvests "below normal," available inventory, \( x \), is below some critical level, one would expect carryover to drop down to zero, since the price now will be too high relative to the price expected for the next period to permit grain that is carried over to earn the interest and shrinkage charges involved in the process: i.e., for \( x < x^* \), \( f(x) = 0 \), and the price inequalities of (6) will hold.

4. In summary, the lower is present price, \( p_t \), the lower the conditional expectation of future prices, but with a quantitative strength that attenuates with time.

Solving \( p = P[x - f(x)] \) inversely for \( x = X[p] \) and \( q = f(X[p]) = q[p] \), we have the following recursion relation connecting \( p_{t+1} \) with \( p_t \) and the intervening random harvest \( p_{t+1} = P[H_{t+1} + aq_{t}] - f(H_{t+1} + aq_{t}) \),
\[ 0 \leq \partial p_{t+1}/\partial p_t \leq 1 \] (14)
From this we can calculate the conditional probabilities
\[ \text{Prob}[p_{t+1} \leq p | p_t = p_t] = \pi_n(p_t; p_0) \]
\[ \text{Prob}[p_{t+1} \leq p | p_t = p_t] = \pi_n(p_t; p_0) \]
\[ \text{Prob}[p_{t+1} \leq p | p_t = p_t] = \pi_n(p_t; p_0) \]
Since \( p_t \) can affect \( p_{t+1} \) or \( p_{t+2} \) only through the effect of its \( q[p_{t+1}] \) on \( p_{t+2} \), we have here a Markov process with
\[ \text{Prob}[p_{t+n} \leq p | p_t = p_t, p_{t+1} = p_{t+1}] = \pi_n(p_t; p_0) \]
Hence, the Markov transitional probabilities will satisfy the Chapman-Kolmogorov relations
\[ \pi_{n+m}(p_r; p_0) = \int_0^\infty \pi_n(p_r; y)d\pi_m(y; p_0) \] (16)
The conditional probabilities of (15) necessarily have the properties
\[ \partial \pi_n(p; p_0)/\partial p_0 \geq 0, \quad \partial E[p_{t+n} | p_0]/\partial p_0 \geq 0 \] (17)
in consequence of (14).
Finally, under weak regularity conditions, an ergodic state is assured, with

$$\lim_{n \to \infty} x_n(p_0) = x(p)$$ independently of $p_0$

$$\lim_{n \to \infty} E[p_{t+n} | p_0] = \int_0^\infty p d\pi(p)$$  \hspace{1cm} (18)

CONCLUSION

A model of (nonindependent) Brownian vibration for stochastic speculative price has been rigorously deduced. It has quantitative properties like those observed for historical organized markets (Board of Trade, Cotton Exchange, etc.), and also broad properties that a technocratic planner would prescribe for well-functioning carryovers. This simple model is readily capable of being generalized—to consider $[H_t]$ subject to a periodic seasonal stochastic fluctuation, to allow serial dependence of harvests in different crop years, to let plantings vary in function of current price (and conditionally expected future prices) by deleting $H_t$ as a variable and making a random variable over the nonnegative range. The model is inadequate to explain the empirical facts of negative carrying charges and "normal backwardation," and therefore needs further amplification.

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