The Characters of the Symmetric Group

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ABSTRACT A short and simple derivation of the formula for Frobenius' irreducible representations of Sn, the symmetric group on any number, n, of symbols, is given. These dimensions are the characters of the identity element of the group, i.e., of the element all of whose cycles are unary. It is shown how a slight modification of Frobenius' formula yields, when n = 2p is even, the character of an element of S2p all of whose cycles are binary and, when n = 3p is a multiple of 3, the character of an element of S3p all of whose cycles are ternary and, generally, when n = kp is a multiple of any positive integer k, the characters of an element of Sn all of whose cycles are of length k. It is noteworthy that the calculations become simpler, rather than more complicated, as k increases. Finally, this paper shows how to derive from Frobenius' formula the characters of an element of Sn which has at least one unary cycle and, from the present modifications of Frobenius' formula, the characters of an element of Sn with at least one cycle of length k, k = 2, 3, . . ., n.

Proof of Frobenius' formula

Almost three quarters of a century have passed since Frobenius gave his much admired formula. Each irreducible representation of Sn is associated with a partition of n; I write any such partition as (n − m, (μ)) where (μ) = (μ1, μ2, . . ., μk) is a j-partition of n and 0 ≤ m ≤ n and it is understood that n − m ≥ μ1 ≥ μ2 ≥ . . . ≥ μj. I arrange these partitions in n in the order (n, 0); (n − 1, 1); (n − 2, 2), (n − 2, 1, 1); (n − 3, 2), (n − 3, 1, 1), and so on, and denote the dimension of the irreducible representation of Sn which is associated with (n − m, (μ)) by dμ(μ). Then I construct the (j + 1) numbers lμ = μ1 − 1, lμ = μ2 − 1, . . ., lμ = μj − 1, μj + 2, . . ., μi = μi + (j − 1), i = n − m + j and observe that lμ > lμ > . . . > lj > lj. Then, Frobenius' formula says that dμ(μ) is the difference product, μ(lμ − lμ), s > r, of the (j + 1) lμ's multiplied by n! and divided by n!lμlμ! . . . lμlμ!. To prove this, I denote by [n − m, (μ)] the characteristic of the irreducible representation of Sn which is associated with the partition (n − m, (μ)) of n and permit the numbers in the symbol [n − m, (μ)] to be reordered, i.e., to be such that one of them is greater than a preceding one, by the following convention: if a and b are two consecutive numbers of the symbol [n − m, (μ)] which are such that b > a, then . . . . a, b, . . . = −[. . . b − 1, a + 1, . . .]. This convention also permits some of the numbers in the symbol [n − m, (μ)] to be reordered, it being understood that if, after it is reordered, the final number is negative the characteristic [n − m, (μ)] vanishes. For example, [2, 4] = [−3, 1] = [−1, 3] = [2] and if, in . . . a, b, c, d, . . ., b = a + 1, or c = a + 2, or d = a + 3, and so on, the characteristic . . . a, b, c, d, . . . vanishes. Also [a1, a2, . . ., ak] vanishes if a1 ≤ a1; thus [−3, 1, 1] vanishes while [−2, 1, 1] = [0] = 1. Of special importance for the present paper is the fact that [n − m, (μ)] vanishes if n − m < −j, i.e., if n has any one of the m − j values, m − j − 1, m − j − 2, . . ., 0. It also vanishes if n has any one of the j values m + μ1 − 1, m + μ2 − 2, . . ., m + μj − j. When a characteristic vanishes all the characters of the irreducible representation of Sn of which it is the characteristic are zero and the dimension of this representation is one of these characters. Thus, the m numbers 0, 1, . . ., m − j − 1, m + μ1 − 1, m + μ2 − 2, . . ., m + μj − j are zeros of dμ(μ), being understood that m > 0, dμ(0) being 1. I proceed to show that, when m > 0, dμ(μ) is a polynomial, of degree m, in n. To do this I introduce the notation (μ) = (μ1 − 1, μ2, . . ., μj + a1 + a2 + . . . + (μj, μj+1, . . ., μk − 1, μk = μ1 − 1). The formula which yields dμ(μ) is dμ(μ) = dμ(μ) = 1 where, if f is any function of n, Δf = f(n + 1) − f(n). Denoting by b0, b1, b2, . . ., the binomial coefficients for n, so that b0 = 1, b1 = n, b2 = n(n − 1)/2, and so on, I have Δdμ(μ) = dμ(μ) = 1 = b0 and so dμ(1) = b1 − b0 the additive constant being determined by the fact that [n − 1, 1] vanishes when n = 1. Also Δdμ(2) = dμ(2) = 1 since dμ(2) = b2 − b1, the additive constant being determined by the fact that [n − 2, 2] vanishes when n = 0. Similarly Δdμ(3) = dμ(1) but, now, dμ(3) = b3 − b2 = b3 − b1 + b2 = b3 + b2 = b1, dμ(4) = b4 − b3 − b2 + b4 = b3 − b2 + b1 = b4 + b3 + b2 + b1, dμ(5) = b5 − b4 − b3 − b2 + b4 = b5 − b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(21) = b5 − b4 − b3 + b2 − b1 = b5 + b4 − b3 − b2 + b1 = b5 − b4 − b3 + b2 + b1 = dμ(31) = b5 − b4 − b3 + b2 + b1 = b5 + b4 − b3 − b2 + b1 = dμ(32) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(31) = b5 − b4 − b3 + b2 + b1 = b5 + b4 − b3 − b2 + b1 = dμ(32) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(312) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(312) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 = b5 + b4 − b3 + b2 + b1 = dμ(321) = b5 + b4 − b3 + b2 + b1 =
c\text{sp}(\mu) is determined by means of the formula \( \Delta c_{\text{sp}}(\mu) = c_{\text{sp}}(\mu) - k \) where, if being any function of \( p, f(p + 1) - f(p) \). 
\( \mu - k \) is zero if \( k > m \) and so \( c_{\text{sp}}(\mu) \) is independent of \( p \) if \( m < k \). More than this, \( c_{\text{sp}}(\mu) \) is independent of \( k \) if \( m < k \), as is seen by setting \( p = 1 \) and using the known characters of an element of \( S_k \) which has just one cycle. Thus \( c_{\text{sp}}(1) = -1, k = 2, 3, 4, \ldots; c_{\text{sp}}(2) = 0, c_{\text{sp}}(1) = 1, k = 3, 4, \ldots; c_{\text{sp}}(3) = 0, c_{\text{sp}}(21) = 0, c_{\text{sp}}(1^2) = -1, k = 4, 5, \ldots; c_{\text{sp}}(4) = 0, c_{\text{sp}}(31) = 0, c_{\text{sp}}(2^2) = 0, c_{\text{sp}}(21^2) = 0, c_{\text{sp}}(1^3) = 1, k = 5, 6, \ldots and so on. If \( k > m \), all \( c_{\text{sp}}(\mu) \) are zero save \( c_{\text{sp}}(1^m) \) which is \( (-1)^m \). Denoting by \( b_0, b_1, \ldots \) the binomial coefficients for \( p \), so that \( b_0 = 1, p, b_1 = p(p - 1)/2 \) and so on, I find the following results, in addition to those just cited: 
When \( k = 2 \), \( c_{\text{sp}}(2) = b_1, c_{\text{sp}}(1^2) = -b_1 + b_0, c_{\text{sp}}(3) = -b_1, c_{\text{sp}}(21) = 0, c_{\text{sp}}(1) = b_1 - b_0 c_{\text{sp}}(4) = b_1, c_{\text{sp}}(31) = -b_1 + b_0, c_{\text{sp}}(2) = 2b_2 - b_1 c_{\text{sp}}(21^2) = -b_1, c_{\text{sp}}(1^2) = b_1 - b_0 b_1 \) and so on. When \( k = 3 \), \( c_{\text{sp}}(3) = b_1, c_{\text{sp}}(31) = -b_1, c_{\text{sp}}(2^2) = b_1, c_{\text{sp}}(21) = -b_1, c_{\text{sp}}(1) = b_1 - b_0 b_1 \) and so on. When \( k = 4 \), \( c_{\text{sp}}(4) = b_1 c_{\text{sp}}(41) = 0, c_{\text{sp}}(32) = -b_1 c_{\text{sp}}(31) = c_{\text{sp}}(2^2) = 0, c_{\text{sp}}(21^2) = b_1 c_{\text{sp}}(21) = 0, c_{\text{sp}}(1^2) = -b_1 + b_0 b_1 \) and so on. When \( k = 5 \), \( c_{\text{sp}}(5) = b_1, c_{\text{sp}}(41) = 0, c_{\text{sp}}(32) = -b_1 c_{\text{sp}}(31) = c_{\text{sp}}(2^2) = c_{\text{sp}}(21^2) = 0, c_{\text{sp}}(1^2) = -b_1 + b_0 b_1 \) and so on. When \( k = 6 \), \( c_{\text{sp}}(6) = b_1, c_{\text{sp}}(51) = 0, c_{\text{sp}}(42) = -b_1 c_{\text{sp}}(41) = -b_1, c_{\text{sp}}(33) = 0, c_{\text{sp}}(2^3) = b_1 c_{\text{sp}}(21^2) = 0, c_{\text{sp}}(1^2) = -b_1 + b_0 b_1 \) and so on. 
Character of an element of \( S_k \) which has cycles of different lengths 

1 and \( k \) unitary cycles, \( k = 2, 3, \ldots, n - 1 \). The characters of such an element are linear combinations of the dimensions of the irreducible representations of \( S_{n-1} \); these combinations being found as follows: \( c(\mu) = c_{\text{sp}}(\mu) + c_{\text{isp}}(\mu - k) = c_{\text{sp}}(\mu) + c_{\text{isp}}(\mu - 1, \mu_2, \mu_3, \ldots) + \ldots + c_{\text{isp}}(\mu_1, \mu_2, \ldots, \mu_k) \). In particular, when \( k > m \), \( c(\mu) = c_{\text{isp}}(\mu) \). When \( k = 2 \), I have \( c(1) = c_{\text{isp}}(1) = n - 3; c(2) = c_{\text{isp}}(2) + c_{\text{isp}}(0) = (n - 3) - (n - 4)/2, c(1') = c_{\text{isp}}(1') = c_{\text{isp}}(0) = (n - 2) - (n - 5)/2; c(3) = c_{\text{isp}}(3) + c_{\text{isp}}(1) = (n - 3) - (n - 4)/2 - (n - 6)/3, c(1'') = c_{\text{isp}}(1'') = c_{\text{isp}}(0) = (n - 4) - (n - 7)/6; c(4) = c_{\text{isp}}(4) + c_{\text{isp}}(2) = (n - 2) - (n - 7)/2 - 9(n - 8)/24, c(1''') = c_{\text{isp}}(1''') = c_{\text{isp}}(0) = (n - 4) - (n - 7)/2 - (n - 8)/3, c(1) = c_{\text{isp}}(1) + c_{\text{isp}}(3) + c_{\text{isp}}(1) = (n - 2) - (n - 7)/2 - (n - 8)/3, c(1) = c_{\text{isp}}(1) + c_{\text{isp}}(3) + c_{\text{isp}}(1) = (n - 2) - (n - 7)/2 - (n - 8)/3, c(1) = c_{\text{isp}}(1) + c_{\text{isp}}(3) + c_{\text{isp}}(1) = (n - 2) - (n - 7)/2 - (n - 8)/3. 

For small values of \( n - k \), I make the usual adjustments of disordered partitions; for example, when \( n = 10, k = 6 \), the 5th representation of \( S_{n} \) is associated with the partition \((13)\) of \( 4 \) and so its character is the negative of the character of the representation which is associated with the partition \((22)\) of \( 4 \).

To deal with an element of \( S_k \) which has, for example, two
binary cycles and $n - 4$ unary cycles, I simply apply the operation "$-2$" twice to $(n - m, (\mu))$; for an element which has one binary, one ternary and $n - 5$ unary cycles, I apply the operations "$-2$" and "$-3$", in sequence, to $(n - m, (\mu))$; and so on. For an element which has no unary cycles, the characters of an element all of whose cycles are binary or ternary, and so on, play the role heretofore played by the characters of an element all of whose cycles are unary. For example, for an element of $S_n$ which has one cycle of length $k$ and $(n - k)/2$ binary cycles the desired characters are linear combinations of the characters of an element of $S_{n-k}$ all of whose cycles are binary, these linear combinations being determined by applying the operation "$-k$" to the partition $(n - m, (\mu))$ of $n$, and so on.