Distribution Function Inequalities for Singular Integrals

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ABSTRACT This paper describes some distribution function inequalities between maximal functions and singular integral operators.

Similar estimates relating the nontangential maximal Poisson integral and the Lusin area function have been proved by D. L. Burkholder and R. F. Gundy [2, 3], who also were the first to recognize their importance for obtaining norm inequalities (these are the so-called good-ω inequalities).

On \( \mathbb{R}^n \) we consider the Hardy–Littlewood maximal operator defined for locally integrable functions by

\[
 f^*(x) = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_{|y| < \varepsilon} |f(x + y)|dy
\]

and the maximal truncated singular integral operator.

\[
 \bar{K}(f)(x) = \sup_{\varepsilon > 0} \left[ \int_{1/\varepsilon < |y| < \varepsilon} |y|^\alpha f(x - y)dy \right]
\]

where \( \Omega(y) \) is a homogeneous function of degree 0 satisfying a Lipschitz condition on the sphere \( \Sigma_{n-1} \) and having mean value 0 on \( \Sigma_{n-1} \). We let \( m(E) \) denote the Lebesgue measure of \( E \).

**Theorem I.** There exist positive constants \( C, C_1, \) such that for all locally integrable functions \( f \) and all \( \beta > 1, \gamma < 1 \) we have

\[
m[\bar{K}(f)(x) > \beta \lambda, f^*(x) \leq \gamma \lambda] \leq C_1 \exp \left( C_2 \frac{1 - \beta}{\gamma} \right) m[\bar{K}(f)(x) > \lambda].
\] (1)

**Corollary I.** Let \( \Phi(t) \geq 0 \) be increasing and such that \( \Phi(2t) \leq C \Phi(t) \) then

\[
 \int_{\mathbb{R}^n} \Phi[\bar{K}(f)(x)]dx \leq C' \int_{\mathbb{R}^n} \Phi[f^*(x)]dx.
\]

Given theorem I, the proof of this corollary follows by an argument of Burkholder and Gundy [2], which we include for completeness.

\[
 \int_0^\infty m[\bar{K}(f) > \lambda]d\Phi(\lambda) \leq \int_{\mathbb{R}^n} \Phi[\bar{K}(f)]dx \leq C \int_{\mathbb{R}^n} \left[ \frac{\bar{K}(f)}{2} \right]dx
\]

\[
 = C \int_0^\infty m[\bar{K}(f) > 2\lambda]d\Phi(\lambda) \leq C \int_0^\infty m[\bar{K}(f) > 2\lambda]
\]

\[
f^* \leq \gamma \lambda d\Phi(\lambda) + C \int_0^\infty m[f^* > \gamma \lambda]d\Phi(\lambda) \leq CCe^{-(C_1/\gamma)}
\]

\[
 \times \int_0^\infty m[\bar{K}(f) > \lambda]d\Phi(\lambda) + C \int_0^\infty m[f^* > \gamma \lambda]d\Phi(\lambda).
\]

Taking \( \gamma = 2^{-N} \) for some large \( N \) and subtracting the first summand in the last expression from both sides, we obtain the result.

The following consequence of theorem I was pointed out to us by R. Gundy.

Let \( \omega(x) \) be locally integrable and

\[
m_\omega(E) = \int_E \omega(x)dx
\]

It was proved by B. Muckenhoupt [ref. 9 and manuscript in preparation] that a necessary condition for the inequality

\[
 \int_{\mathbb{R}^n} |f^*|^2 \omega(x)dx \leq C \int_{\mathbb{R}^n} |f|^2 \omega(x)dx
\]

to hold is that there exist \( C > 0, \delta > 0 \) such that for all balls \( B \) in \( \mathbb{R}^n \) and measurable sets \( E \).

\[
 \frac{m_\omega(E \cap B)}{m_\omega(B)} \leq C \left[ \frac{m(E \cap B)}{m(B)} \right]^\delta
\] (2)

It is an immediate consequence of the proof of theorem I that for such weights we have

\[
m_\omega[\bar{K}(f) > \beta \lambda, f^* \leq \gamma \lambda] \leq C \exp \left( C_1 \frac{1 - \beta}{\gamma} \right) m_\omega[\bar{K}(f) > \lambda]
\]

from which one obtains the corresponding norm inequalities.*

We would like to point out that although the results stated involve the Hardy–Littlewood maximal function, they can be considerably refined to include "smoother" versions of \( f^* \), such as the ones used by Fefferman and Stein [5]. The main tool used is their analogue of the Calderón–Zygmund lemma. In order to explain the idea of the proof, I will give the argument in the case of the Hilbert transform for a slightly weaker form of inequality (1).

Let

\[
 f(x) = \sup_{\varepsilon \in \epsilon} \left[ \int_{1/\varepsilon < |y| < \varepsilon} \frac{f(t)}{|x - t|} dt \right]
\]

we shall show that there exist constants \( C, C' > 0 \) such that;

\[
m[f(x) > \beta \lambda, f^*(x) \leq \gamma \lambda] \leq \frac{C \gamma}{\beta - 1 - \gamma C} m[f(x) > \lambda].
\] (3)

We can assume \( m[f(x) > \lambda] < \infty \). Since \( E_\lambda = \{ f(x) > \lambda \} \) is open, we must have \( E_\lambda = \bigcup I_i \) with \( I_i = (\alpha_i, \alpha_i + \delta_i) \) dis-

* This provides a real variable proof of the result of Helson and Szego concerning weighted \( L^2 \) inequalities for conjugate functions [6]; see also R. Hunt, B. Muckenhoupt, and R. Wheeden [8].
joint intervals and $f(\alpha_i) \leq \lambda$. It is sufficient to prove that, for each $i$,

$$m[x \in I_i; f(x) > \beta \lambda, f^*(x) \leq \gamma \lambda] \leq \frac{C m(I_i)}{\beta - 1 - \gamma C'} \ldots \ (4)$$

Clearly we can also assume that there is a point $\xi \in I_i$ for which $f^*(\xi) \leq \gamma \lambda$. Let $I_i = (\alpha_i - 2\delta_i, \alpha_i + 2\delta_i)$ and define

$$f_1(x) = \begin{cases} f(x) & x \in I_i \\ 0 & x \notin I_i \end{cases}$$

We claim:

(a) $\forall \beta' > 0 \ m[x \in I_i; f_1(x) > \beta' \lambda] \leq \frac{C m(I_i) \lambda}{\beta'}$

(b) $\forall x \in I_i, f_1(x) \leq (C' \gamma + 1) \lambda$.

In fact, to see (a) we observe that the maximal truncated Hilbert transform is of weak type $1 - 1; \lambda$ that is,

$$m[x \in I_i; f_1(x) > \beta \lambda] \leq C \int \frac{f_1(x)dx}{\beta \lambda} =$$

$$= \frac{C 4 m(I_i)}{\beta \lambda} \left[ \int \frac{f_1(x)dx}{m(I)} \int f(x)dx \right] \leq \frac{C \gamma m(I_i)}{\beta', \gamma}$$

the last inequality follows from the assumption $f^*(\xi) \leq \gamma \lambda$ for some $\xi \in I_i$.

For the proof of (b), we write, for $x \in I_i$,

$$\int_{1/\alpha_i < |x-t| < \alpha_i} \frac{f_1(x)}{x-t} dt = \int \frac{f_1(x)}{x-t} dt$$

$$+ \int \frac{f_1(x)}{x-t} \left( \frac{1}{x-t} \right) dt$$

$$+ \left[ \int_{1/\alpha_i < |x-t| < \alpha_i} \frac{f_1(x)}{x-t} - \int_{1/\alpha_i} \frac{f_1(x)}{x-t} dx \right]$$

The first term is less than $\lambda$ (by construction) and the last two terms are easily seen to be dominated by $C f^*(\xi)$ for any $\xi \in I_i'$ taking $\xi$ such that $f^*(\xi) \leq \gamma \lambda$ we obtain inequality (b).

We now let $\beta' = (\beta - 1 - \gamma C')$. Then, using (b) and then (a)

$$m[x \in I_i; f(x) > \beta \lambda, f^*(x) \leq \gamma \lambda] \leq m[x \in I_i; f_1(x) + \frac{f(x)}{\lambda} > \lambda \beta' + (1 + \gamma C') \lambda] \leq m[I_i; \frac{f(x)}{\lambda} > \lambda \beta']$$

$$\leq \frac{C m(I_i)}{\beta'} \gamma = \frac{C m(I_i)}{\beta - 1 - \gamma C'}.$$  

This proves (4) and, thus, (3) follows.

We now would like to explain some of the technical modifications needed in order to obtain (1). To obtain the exponential estimate we need only use, in (a), the result of R. Hunt [7] or, if we wish, the Calderón-Zygmund decomposition of $f_1$, combined with the usual exponential estimate for singular integrals of bounded functions. The extension of the proof to $\mathbb{R}^n$ involves only the use of Whitney's lemma and standard estimates.

To conclude, we would like to point out that a refinement of the same proof yields the following general result.

Let $k(x,y)$ be such that $\int k(x,y)f(y)dy$ is a bounded operator on $L^1(\mathbb{R}^n)$ and $k(x,y)|x-y|^1 \leq 1$, $|k(x,y) - k(x,y_0)| \leq |y - y_0|^{-n}$, $|y - y_0| > 2|y - y_0|$ and $|k(x,y) - k(x,y_0)| \leq |y - y_0|^{-1}|y - y_0| > 2|y - y_0|$ then

$$R(f)(x) = \sup_{\epsilon \rightarrow 0} \left| \int_{|t - x|^n > \epsilon} k(x,t)f(y)dy \right|$$

is a bounded operator on $L^1(\mathbb{R}^n)$ and

$$m(R(f) > \beta \lambda, f^* < \gamma \lambda) \leq C \exp \left( C \frac{\gamma - \beta}{\gamma} \right) m(I)$$

An example for such operators is provided by the commutator singular integral of Calderón

$$C(f)(x) = \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{(x - y)^2} f(y) dy$$

where $|a(x) - a(y)| \leq |x - y|$ and their n-dimensional analogues see refs. 1 and 4.

We also claim that a more general result is true for this class of operators. For every $s > 1$ there exist constants $C, C'$ depending only on $s$ such that

$$m(C(f) > \beta \lambda, M_s(a')(x) f^*(x) \leq \gamma \lambda)$$

$$\leq C \frac{\gamma - \beta}{\gamma} m(C \lambda)$$

$$\leq C \frac{\gamma - \beta}{\gamma} m(C \lambda)$$

where $M_s(a')(x) = ||(a'(x))||^s_{1,1}$. This implies (as before)

$$\int_{-\infty}^{\infty} \Phi[C(f)(x)] dx \leq C \int_{-\infty}^{\infty} \Phi[M_s(a')(x) f^*(x)] dx$$

which, even for the case $\Phi(t) = t^r, r > s$, is a refinement of Calderón’s result in [4].

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