Correction. In the article "Stationary Spherical Vortices in a Perfect Fluid," by C. L. Pekeris, which appeared in the September 1972 issue of the Proc. Nat. Acad. Sci. USA 69, 2460–2462, on p. 2460, Eq. [18] should read: \[ \nabla \times \mathbf{U} = \lambda \tau C \frac{1}{\sin \theta} \frac{\partial Y}{\partial \phi} 1_\theta + \lambda \tau C \frac{\partial Y}{\partial \phi} 1_\phi, \quad n = 1, \]

where \( C \) is the constant appearing in Eq. [51]. The Bernoulli equation takes the form

\[ (p/\rho) + \Omega = -\frac{1}{2} (u^2 + v^2 + w^2) + \lambda \tau CS \times (1 + A^2 - Y_i^2), \quad n = 1, \]

\( A \) denoting the constant in Eq. [26].

Correction: In the article "Reactions Involved in Bioluminescence Systems of Limpet (Lottia neritoides) and Luminous Bacteria," by Shimomura, O., Johnson, F. H. & Kohama, Y., which appeared in the August 1972 issue of the Proc. Nat. Acad. Sci. USA 69, 2086–2089, in the Abstract (p. 2086, line 13), "0.17 + 0.01 photons" should read: "0.17 ± 0.01 photons." On page 2088, right-hand column, line 9 from top, "0.154 ± 0.1 einstein/mol" should read: "0.154 ± 0.01 einstein/mol"; and, same page and column, line 13 from top, "0.17 ± 0.1" should read: "0.17 ± 0.01."
Stationary Spherical Vortices in a Perfect Fluid

(incompressibility/flow/trajectory/dynamo theory/stationarity conditions)

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ABSTRACT Necessary conditions are derived for a spherical vortex in a perfect fluid to be stationary in the case when the velocities depend on a single surface harmonic. The motion is indeterminate unless an additional condition is imposed. In the case when this condition is incompressibility, the equations are solved, yielding a class of stationary spherical vortices.

In this investigation, I wish to determine the types of stationary vortex motion that are possible in a fluid that is bounded by spherical surfaces. The condition of stationarity imposes a restriction on the pattern of flow (1), since the streamlines have to coincide with the trajectories of the fluid particles. The immediate application I have in mind is to the dynamo theory of the origin of the earth's magnetic field, proposed by Larmor (2). In the stationary kinematic dynamo (3), the magnetic field \( \mathbf{H} \) is determined by the equation

\[
\nabla \mathbf{H} + V \text{curl}(\mathbf{U} \times \mathbf{H}) = 0,
\]

where \( \mathbf{U} \) is a given convective field inside the liquid core of the earth. Here, \( V \) is a nondimensional eigenvalue given by

\[
V = 4\pi \bar{b} U_0,
\]

where \( \bar{b} \) is the electrical conductivity of the liquid core, \( \bar{b} \) is the radius of the core, and \( U_0 \) the velocity scale of the convection.

In deciding on the type of convective field to choose, I shall assume as a working hypothesis that if a certain stationary convective cell is possible in the case of a perfect fluid, such a pattern of flow is likely to be established also in a real fluid, subject to local modifications by dissipative forces.

NECESSARY CONDITIONS FOR A STATIONARY FLOW

In a spherical system of coordinates \((r, \theta, \phi)\) with velocity components \((u, v, w)\) the stationary motion has to satisfy the Eulerian equations

\[
\frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial u}{\partial \phi} = -\frac{1}{r} \frac{\partial Q}{\partial r},
\]

\[
\frac{\partial v}{\partial r} + \frac{u}{r} \frac{\partial v}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial v}{\partial \phi} + \frac{w}{r} \cotan \theta = -\frac{1}{r \sin \theta} \frac{\partial Q}{\partial \phi},
\]

where

\[
Q = \frac{p}{\rho} + \Omega,
\]

\( p \) denoting the pressure, \( \rho \) the density, and \( \Omega \) the potential of the forces. I shall restrict the discussion to the case where the motion depends on a single spherical surface harmonic

\[
Y = Y_\nu^m = \sum_{n=0}^{\infty} (A_n^m \cos \phi + B_n^m \sin \phi) P_n^m (\cos \theta),
\]

through (4),

\[
\begin{align*}
u &= U(r) Y, \\
v &= V(r) (\partial Y / \partial \theta) + [W(r) / \sin \theta] (\partial Y / \partial \phi), \\
w &= [V(r) / \sin \theta] (\partial Y / \partial \phi) - W(r) (\partial Y / \partial \theta).
\end{align*}
\]

Here, \( U(r) \) and \( V(r) \) represent the poloidal field, and \( W(r) \) the toroidal field. The dilatation is given by

\[
\text{div} \mathbf{U} = X(r) Y,
\]

\[
X = \dot{U} + (2/r) U - n(n + 1) V/r,
\]

the dot denoting differentiation with respect to \( r \). In the case of the bodily tides, the motion is poloidal (5) in the absence of rotation. The rotation of the earth induces a toroidal component.

Substitution of [8], [9], and [10] into [3], [4], and [5] yields

\[
-\frac{\partial Q / \partial \theta} = U \dot{Y}^2 + (1/r)(UV - V^2 - W^2),
\]

\[
-\frac{\partial Q / \partial \phi} = \frac{1}{\sin \theta} \frac{\partial L}{\partial \theta} + \frac{MY(1/\sin \theta) (\partial Y / \partial \phi)}{r},
\]

\[
-\frac{\partial Q / \partial \phi} = \frac{1}{\sin \theta} \frac{\partial L}{\partial \theta} - M \sin \theta Y (\partial Y / \partial \phi),
\]

where

\[
L = \left\{ \left[ U(r \dot{V} + V) + n(n + 1) W^2 \right] \dot{Y}^2 \\
+ (V^2 + W^2) \dot{Y} \right\},
\]

\[
M = \left[ U(r \dot{W} + W) - n(n + 1) V W \right],
\]

\[
\dot{Y} = \left[ (\partial Y / \partial \theta)^2 + (1/\sin^2 \theta) (\partial Y / \partial \phi)^2 \right].
\]

I shall assume, in the first instance, that \( Y \) in [7] depends on \( \phi \). By equating the expressions for \( \partial Q / \partial \phi \) derived from [14] and from [15], we get

\[
M \cdot [-n(n + 1) Y^2 + \dot{Y}] = 0.
\]

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Similarly, by crossdifferentiating [13] and [14], we get
\[ NY(\partial Y/\partial \phi) + P(\partial Y/\partial \phi) - \dot{M}(1/\sin \theta) Y(\partial Y/\partial \phi) = 0, \]  
where
\[ N = (1/d/dr)(U^2 - U(r\dot{V} + V) - n(n + 1)W^2), \]  
\[ P = \left[ (1/r)(UV - V^2 - W^2) - \dot{V} - W\dot{W} \right]. \]

The expressions of \((\partial^2 Y/\partial \phi^2)\) yield, by [13] and [15],
\[ NY(\partial Y/\partial \phi) + P(\partial Y/\partial \phi) + \dot{M} \sin \theta Y(\partial Y/\partial \phi) = 0. \]

Conditions [19], [20], and [23] are necessary for a perfect fluid in order that the spherical vortex represented by [8], [9], and [10] shall be stationary. Since the expression in brackets in Eq. [19] does not vanish for any spherical surface harmonic \(Y\), we must have
\[ M = U(r\dot{W} + W) - n(n + 1)VW = 0. \]

It follows from [20] and [23] that
\[ NY^2 + 2P\dot{Y} = f(r). \]

Condition [25] takes on a special form in the case \(n = 1\):
\[ Y = Y_1 = \cos \theta + Asin\theta \cos(\phi - \phi_0), \]
when the following relation holds:
\[ Y_1^2 + \dot{Y}_1 = 1 + A^2. \]

For the case \(n = 1\), [25] therefore requires that
\[ N_1 - 2P_1 = 0, \]
\[ 2P(1 + A^2) = f(r). \]

For \(n > 1\), condition [25] requires, in addition to the vanishing of \(f(r)\), that
\[ N = 0, \]
\[ P = 0. \]

Eq. [30] integrates into
\[ U(r\dot{Y} + V) + n(n + 1)W^2 - U^2 = 0, \]
while [31] can be put in the form
\[ UV - V(r\dot{Y} + V) - W(r\dot{W} + W) = 0. \]

The three Eqs. [24], [32], and [33] do not suffice to determine the functions \(U, V, \) and \(W\), because they are not independent:

Therefore, we can take two of them, say [24] and [32], as the necessary conditions for the existence of a steady motion of the type represented by Eqs. [8], [9], and [10].

Eliminating \(V\) from [12] and [24], we get
\[ r\dot{W} = [\lambda/n(n + 1)]r^2U \exp \left[ -\int (X/U)dr \right], \]
which is valid also for the case \(n = 1\). Here, \(\lambda\) is an arbitrary constant. Substituting from [12] and [35] into [32], we arrive at
\[ (d^2/dr^2)(r^2U) - n(n + 1)U + \lambda r^2 UG^2 \]
\[ = (d/dr)(r^2X), \]
where
\[ G(r) = \exp \left[ -\int (X/U)dr \right]. \]

In the case \(n = 1\), Eq. [28] can be put in the form
\[ (d/dr)ln[U^2 - U(r\dot{V} + V) - 2W^2] = (d/dr)ln(rW). \]

Hence,
\[ U^2 - U(r\dot{V} + V) - 2W^2 = ArW. \]

By [12] and [35], Eq. [39] yields
\[ (d^2/dr^2)(r^2U) - 2U + \lambda r^2 UG^2 = (d/dr)(rX) + KrG. \]

Let (3)
\[ r^2U(r) = n(n + 1)S(r), \]
then
\[ V = (\dot{S}/r) - [r^2X/(n(n + 1))], \]
\[ W = (\lambda/r)S. \]

Eqs. [36] and [40] then take on the form
\[ \ddot{S} + [\lambda G^2 - (n(n + 1)/r^2)] S = 1/n(n + 1) \]
\[ \times (d/dr)(rX), \]
\[ \ddot{S} + [\lambda G^2 - (2/r^2)] S = 1/4(d/dr)(rX) + Kr^2G, \]

**CASE OF AN INCOMPRESSIBLE FLUID**

In the case of incompressibility, we have
\[ X = 0, \]
\[ G = 1, \]
\[ V = (\dot{S}/r), \]
\[ W = (\lambda/r)S. \]

The solution of [48] is
\[ S_n = A(\lambda r)^{1/2}J_{n+1/2}(\lambda r) + B(\lambda r)^{1/2}J_{n-1/2}(\lambda r), \]
\[ n > 1, \]
while the solution of [49] is
\[ S_1 = A(\lambda r)^{1/2}J_{1/2}(\lambda r) + B(\lambda r)^{1/2}J_{-1/2}(\lambda r) + Cr^2. \]

To summarize, in the case of an incompressible ideal fluid there exists a class of stationary spherical vortices defined by
\[ u = [n(n + 1)/r^2]S(r)Y, \]
\[ v = (\dot{S}/r)(\partial Y/\partial \phi) + (T/r \sin \theta)(\partial Y/\partial \phi), \]
\[ w = (\dot{S}/r \sin \theta)(\partial Y/\partial \phi) - (T/r)(\partial Y/\partial \phi), \]
where \(Y\) is a surface spherical harmonic defined in [7],
\[ T(r) = \lambda S(r), \]
and \(S(r)\) is given by Eqs. [50] and [51].
**THE HICKS SPHERICAL VORTEX**

The special case of an incompressible ideal fluid where the motion possesses axial symmetry was treated by Hicks (6). In this case of axial symmetry (7), $M(r)$ must vanish by [15], and with it Eq. [23] is satisfied. The remaining condition [20] takes the form

$$
(\partial Y/\partial \theta) \left( N - 2n(n + 1)P \right) Y - 2P \cot \theta (\partial Y/\partial \theta) = 0.
$$

Condition [56] requires that [28] be satisfied in case $n = 1$, and that [30] and [31] be satisfied for $n > 1$.

**SOLUTION FOR THE PRESSURE**

In order to solve for the pressure, we note that, by virtue of [24], a solution of [14] and [15] is

$$
Q = -u^2 L + C = C - \frac{1}{2} \left( U^2 Y^2 + (V^2 + W^2) Y \right)
= -\frac{1}{4} \left( u^2 + v^2 + w^2 \right) + C,
$$

where use has been made of [32]. The solution [57] satisfies [13] by virtue of [32]. Substituting from [6], we get a Bernoulli equation

$$
(p/\rho) + \Omega = -\frac{1}{4} \left( u^2 + v^2 + w^2 \right) + C.
$$

The constant $C$ in [58] is an absolute constant, and does not differ from one streamline to another (8).

A characteristic feature of our spherical vortices, defined by equations [52]-[55], is that their vortex lines coincide with the streamlines:

$$
\nabla \times \mathbf{U} = \lambda \mathbf{U}.
$$

It follows from the hydrodynamic equation that

$$
(\nabla \times \mathbf{U}) \times \mathbf{U} = -\nabla \left[ \frac{p}{\rho} + \frac{1}{2} \left( u^2 + v^2 + w^2 \right) \right] = 0,
$$

hence relation [58].

Taking the curl of [59] we get

$$
\nabla \times (\nabla \times \mathbf{U}) = \nabla^2 \mathbf{U} = \lambda^2 \mathbf{U},
$$

of which [52]-[55] are solutions.

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