Unipotent Groups in Invariant Theory

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ABSTRACT. The finite generation of the ring of invariants of a special class of unipotent groups is established—namely, unipotent radicals of parabolic subgroups.

Throughout, $F$ will stand for a fixed base field of characteristic 0. By an affine algebraic $F$-group we mean the group of $F$-rational points of an affine algebraic group defined over $F$. We are concerned with rational actions, by $F$-algebra automorphisms, of such groups on finitely generated commutative $F$-algebras. Let $F'$ denote an algebraic closure of the field $F$. For any affine algebraic $F'$-group $H$, we let $H'$ denote the affine algebraic $F'$-group obtained from $H$ in the natural fashion by extension of the base field from $F$ to $F'$ (in the more usual language, we have $H = G_p$, where $G$ is an affine algebraic group defined over $F$, and $H' = G_{p'}$). In these terms, the theorem we shall present here is as follows.

THEOREM. Let $F$ be a field of characteristic 0, and let $G$ be a connected semisimple affine algebraic $F$-group. Let $R$ be a finitely generated commutative $F$-algebra, and suppose there is given a (locally finite) rational $G$-module structure on $R$ by which the elements of $G$ act as $F$-algebra automorphisms. Let $U$ be a unipotent algebraic subgroup of $G$, and suppose that $U'$ is the unipotent radical of some algebraic subgroup of $G'$ containing a maximal unipotent subgroup of $G$. Then the $U'$-fixed part $R'^U$ of $R$ is finitely generated as an $F$-algebra. In particular, this conclusion holds for every maximal unipotent subgroup $U$ of $G$.

Before proceeding to the proof, we remark that the assumptions made on $U$ for the first part of the theorem are evidently satisfied when $U'$ is the unipotent radical of a parabolic subgroup of $G'$. This serves to justify the last statement of the theorem, as follows. Let $U$ be a maximal unipotent subgroup of $G$. By ref. 1, Proposition 8.4, p. 124, $U'$ is then the unipotent radical of some parabolic subgroup of $G'$ (in fact, of one that is minimal among those of the form $P'$, with $P \subseteq G$).

Let $R' = R \otimes F'$. The given rational $G$-module structure of $R$ extends naturally to a rational $G'$-module structure of $R'$ by which the elements of $G'$ act as $F'$-algebra automorphisms.

It is easy to see from standard elementary facts that $R'^U$ coincides with $R' \otimes F'^U$. Therefore, it will suffice to prove that $R'^U$ is finitely generated as an $F'$-algebra. In other words, no generality is lost in assuming that $F$ is algebraically closed.

The next proposition serves to reduce the theorem to the special case where $U'$ is a maximal unipotent subgroup of $G'$, but its force goes beyond this application. For use from now on, we recall the following well-known facts. If an affine algebraic $F$-group $G$ acts on a finitely generated $F$-algebra $R$ as described in the statement of the theorem, let us say simply that $R$ is a rational $G$-module algebra. If $G$ is reductive then, because $F$ is of characteristic 0, $G$ is linearly reductive, in the sense that every rational $G$-module is semisimple. The classical invariant theorem says that, if $G$ is linearly reductive, then $R^G$ is finitely generated as an $F$-algebra (ref. 2, Theorem 5.9, p. 160).

PROPOSITION. Let $H$ be an affine algebraic $F$-group, and let $R$ be a rational $H$-module algebra. Let $H_u$ denote the unipotent radical of $H$, and let $M$ be a unipotent algebraic subgroup of $H$ containing $H_u$. Then $R^{H_u}$ coincides with the smallest $H$-submodule $H \cdot R^M$ of $R$ that contains $R^M$. It follows that, if $R^M$ is finitely generated as an $F$-algebra, so are $R^{H_u}$ and $R^H$.

Proof: Clearly, $R^M \subset R^{H_u}$. Since $H_u$ is normal in $H$, we have $H \cdot R^{H_u} \subset R^H$, Hence $H \cdot R^M \subset R^H$. In order to prove the reversed containment, note that the action of $H$ on $R^{H_u}$ factors through the reductive group $H/H_u$, whence $R^{H_u}$ is semisimple as an $H$-module. Therefore, it suffices to show that every simple $H$-submodule, $V$, say, of $R^{H_u}$ is contained in $H \cdot R^M$. If $V$ is not (0) then we have $V^M \neq (0)$, because $M$ is unipotent. A fortiori, we have then $V \cap (H \cdot R^M) \neq (0)$. But this is an $H$-submodule of $V$, so that it must coincide with $V$, i.e., $V \subset H \cdot R^M$. Since the action of $H$ on $R$ is locally finite, $H \cdot R^M$ is finitely generated as an $F$-algebra whenever this is true for $R^M$. This completes the proof of the proposition, because the conclusion for $R^M$ follows from the conclusion for $R^{H_u}$ by the classical invariant theorem.

We have seen already that it will suffice to prove the theorem in the case where $F$ is algebraically closed. The required result then follows immediately from the proposition once we have proved the following: let $G$ and $R$ be as in the statement of the theorem, and suppose that $F$ is algebraically closed. Let $U$ be a maximal unipotent subgroup of $G$. Then $R^U$ is finitely generated as an $F$-algebra.

We convert this into a Lie algebra problem as follows. Let $L$ be the Lie algebra of $G$, and let $N$ be the Lie algebra of $U$. Then $L$ is semisimple, and $N$ is a maximal ad-nilpotent sub Lie algebra of $L$. As is well known from the theory of semisimple Lie algebras over an algebraically closed field of characteristic
O, there is a Cartan subalgebra $H$ of $L$ and an ordering of the roots of $L$ with respect to $H$ such that $N = \sum_{\alpha<0} L_{\alpha}$, where $L_{\alpha}$ denotes the root subspace of $L$ belonging to the root $\alpha$. Recall that if a connected affine algebraic $F$-group $T$ acts rationally by $F$-algebra automorphisms on an $F$-algebra $A$ then its Lie algebra acts by $F$-algebra derivations on $A$, and the $T$-fixed part of $A$ coincides with the part of $A$ that is annihilated by the Lie algebra of $T$. Hence, it will suffice to prove the following: Let $L$ be a finite-dimensional semisimple Lie algebra over $F$. Suppose that $L$ is split over $F$, in the sense that it has a Cartan subalgebra $H$ such that the roots of $L$ with respect to $H$ take their values in $F$ (of course, this is no restriction in the case where $F$ is algebraically closed). Let $N$ be a sub Lie algebra of $L$ of the form $\sum_{\alpha<0} L_{\alpha}$, as above. Let $R$ be a finitely generated commutative $F$-algebra that is endowed with the structure of a locally finite $L$-module such that the elements of $L$ act as $F$-algebra derivations on $R$. Then the $N$-annihilated part $R^N$ of $R$ is finitely generated as an $F$-algebra.

In order to prove this, we begin with a recall of some of the standard facts from the representation theory of split semisimple Lie algebras. Everything we use from that theory can be found in ref. 3. There is a basis $(h_1, \ldots, h_r)$ of $H$ with the following property. Let $\gamma_i$ denote the linear function on $H$ given by $\gamma_i(h_j) = 1$ and $\gamma_i(h_j) = 0$ for every $j \neq i$. Then the highest weights of the finite-dimensional simple $L$-modules are precisely the linear combinations, with non-negative integer coefficients, of the linear functions $\gamma_1, \ldots, \gamma_r$. The isomorphism class of a finite-dimensional simple $L$-module $V$ is uniquely determined by its highest weight, and $V$ is the direct $H$-module sum of its weight subspaces. The highest and the lowest weight subspace of $V$ are each of dimension 1. If $\gamma$ is a weight of $V$, and if $V_\gamma$ denotes the corresponding weight subspace of $V$, then $N \cdot V_\gamma$ is contained in the sum of the weight subspaces $V_\delta$ with $\delta < \gamma$. Finally, the $N$-annihilated part of $V$ is precisely the lowest weight subspace.

Now let $V_i$ be a simple finite-dimensional $L$-module with highest weight $\gamma_i$. Let $V$ be the direct sum of the $L$-modules $V_1, \ldots, V_r$. Put $P = R \otimes S[V]$, where $S[V]$ is the symmetric algebra built over $V$. For each $i = 1, \ldots, r$, let us choose a basis $\{f(i,1), \ldots, f(i,d_i)\}$ of $V_i$ such that each $f(i,j)$ lies in a weight subspace of $V_i$ and $f(i,1)$ spans the highest weight subspace. We let $L$ act on $P$ by $F$-algebra derivations in the natural way, given the action on $L$. Let $N$ be the $L$-module structure of $V$. As an $R$-algebra, $P$ is the polynomial algebra $R[f(1,1), \ldots, f(r,d_r)]$. We define a surjective $R$-algebra homomorphism $\eta: P \to R$ by setting $\eta(f(i,1)) = 1$ for each $i$, and $\eta(f(i,j)) = 0$ whenever $j \neq 1$.

We know that each $N \cdot V_i$ is contained in the sum of the weight subspaces of $V_i$ that belong to weights other than the highest one. Hence, the elements of $N \cdot V_i$ are $F$-linear combinations of the $f(i,j)$'s with $j \neq 1$. It is clear from this that $\eta$ is also a homomorphism of $N$-modules, whence $\eta(P^L) \subseteq R^N$. We shall prove that, actually, $\eta(P^L) = R^N$.

Since $R$ is locally finite and semisimple as an $L$-module, every element of $R^N$ is a finite sum of elements of $R^N$ of which belongs to some finite-dimensional simple $L$-submodule of $R$. Therefore, it suffices to prove the following: let $U$ be a finite-dimensional simple $L$-submodule of $U$, and let $u$ be an element of $U^N$. Then there is an element $p$ in $P^L$ such that $\eta(p) = u$. As we have noted in the beginning of this proof, $u$ belongs to the lowest weight subspace $U_{\mu}$ of $U$. Let us choose a basis $(u_1, \ldots, u_n)$ of $U$ such that each $u_i$ lies in a weight subspace $U_{\mu}$ and $u_i$ spans the lowest weight subspace, so that $\mu_1 = \mu$ and $u$ lies in $F(u)$. Let $U^0$ denote the $L$-module dual to $U$, and let $(u_1^*, \ldots, u_n^*)$ be the basis dual to $(u_1, \ldots, u_n)$. Then each $u_i^*$ belongs to a weight subspace of the simple $L$-module $U^0$, and the weight of $u_i^*$ is $-\mu_i$. The highest weight of $U^0$ is $-\mu$, so that there are non-negative integers $n_i$ such that $-\mu = \sum n_i \gamma_i$.

Now $S[V]$ contains $S^n[V_1] \otimes \cdots \otimes S^n[V_r]$, and the highest weight of this tensor product is precisely $-\mu$, so that this tensor product contains a $L$-simple component $W$ isomorphic with $L^0$, whose highest weight subspace is spanned by $f(1,1)^n \cdots f(r,1)^{n_r}$ (the multiplication here is that of the algebra structure of $P$). Now we note that $U \otimes U^0$ contains the $L$-annihilated element $u_1 \otimes u_1^0 + \cdots + u_n \otimes u_n^0$ [which is the element corresponding to the identity map $U \to L$ under the standard $L$-module isomorphism $U \otimes U^0 \to \text{Hom}(U, U)$]. The weight of $u_1^0$ is equal to the weight of $f(1,1)^n \cdots f(r,1)^{n_r}$, and each $f_i$ other than 1, the weight of $u_i^0$ is strictly smaller than that of $u_i^0$. Replacing each $u_i$ with its image in $W$ under an $L$-module isomorphism $U \to W$, we obtain an element $p_1$ of $P^L$. From what we have just remarked concerning the weights of the $u_i^0$'s, it is clear that $p_1$ is of the form

$$c u_1 f(1,1)^n \cdots f(r,1)^{n_r} + p_1,$$

where $c$ is a non-zero element of $F$, and $p_1$ belongs to the ideal of $P$ that is generated by the $f(i,j)$'s with $j \neq 1$. This shows that $\eta(p_1) = cu_1$, which spans $U_{\mu}$. Thus, our original element $u$ of $R^N$ is an $F$-multiple of $\eta(p_1)$, and we have the required element $p$ of $P^L$ as some $F$-multiple of $p_1$.

Now we have shown that $\eta(P^L) = R^N$. Since $L$ is semisimple, it is known from the classical invariant theorem that $P^L$ is finitely generated as an $F$-algebra. Hence, the same is true for its homomorphic image $R^N$. This completes the proof of our theorem.

We observe that the last part of the theorem has an almost immediate extension, as follows. If $G$ and $R$ are as in the theorem, and if $V$ is any algebraic subgroup of $G$ containing a maximal unipotent subgroup $U$ then $R^U$ is finitely generated as an $F$-algebra. In order to see this, we note first that, in view of the invariant theorem for finite groups, we may assume that $V$ is connected. Now we apply ref. 1, Proposition 8.4, p. 124, which says that there is a parabolic subgroup $P$ of $V'$ such that $V' = P_u$. The action of $P$ on $F^0$ factors through the reductive group $P'/U'$. We know from the theorem that $R^U$ is finitely generated as an $F$-algebra, whence $R^P$ is finitely generated as an $F$-algebra. Therefore, the classical invariant theorem shows that $(R^P)^{P'/U'}$, i.e., $R^P$, is finitely generated as an $F$-algebra. From the fact that $V'/P'$ is a projective variety, we know that the only everywhere defined rational functions on $V'/P'$ are the constants. Applying this to the relevant representative functions associated with the representation of $V'$ on $R'$, we see that $R^P = R^{V'}$, whence we have the desired result.

The theorem may be regarded as an explanation of what lies hidden under an ingenious device contained in C. Seshadri's proof of the following result. Let $F^+$ denote the additive group of $F$, regarded as a unipotent affine algebraic $F$-group. Let $V$ be a finite-dimensional rational $F^+$-module, and view the symmetric algebra $S[V]$ as a rational $F^+$-module algebra in the natural fashion. Then $S[V]^{F^+}$ is finitely generated as an $F$-algebra. For the details of Seshadri's proof, and for the history of the result, we refer the reader to ref. 4 and ref. 2, pp.
The following outline (made with the help of ref. 2) will clarify the connection with our theorem.

First, one observes that the action of $F^+$ on $V$ can be extended to an action of the special linear group $SL(2,F)$ on $V$, with $F^+$ imbedded in $SL(2,F)$. Now $SL(2,F)$ acts by $F$-algebra automorphisms on $S[V]$. Next, Seshadri adjoins two indeterminates $x$ and $y$ to obtain an enlarged setting in the algebra $S[V][x,y] = S[V + W]$, where $W = Fx + Fy$ and is viewed as an $SL(2,F)$-module by the identity representation of $SL(2,F)$. Then he shows that the $F$-algebra homomorphism $S[V][x,y] \rightarrow S[V]$ obtained from the specialization $x \rightarrow 1$, $y \rightarrow 0$ induces a surjective $F$-algebra homomorphism

$$S[V][x,y]^{SL(2,F)} \rightarrow S[V]^{F^+}.$$  

Here, Seshadri's proof uses algebraic–geometric machinery and fails to reveal the fact that this is easily obtainable from the representation theory of $SL(2,F)$ alone.

In the terms of our proof of the Lie algebra version, the situation of Seshadri's theorem is the following. The Lie algebra $L$ is the Lie algebra of $SL(2,F)$. This is the split 3-dimensional simple Lie algebra, with canonical basis $(h,x,y)$, where $[h,x] = 2x$, $[h,y] = -2y$, and $[x,y] = h$. The imbedding of $F^+$ in $SL(2,F)$ is chosen so that the corresponding sub Lie algebra is $Fy$. In the notation of our proof, one has $V = V_1$, and $V_1$ is of dimension 2. Using the well-known models for the finite-dimensional simple $L$-modules (one for each dimension), one can exhibit the element $p$ of our proof explicitly. In fact, if the dimension of $U$ is $n + 1$, one may write $u = y^n u'$, where $u'$ lies in the highest weight subspace of $U$, so that $h \cdot u' = nu'$. The 2-dimensional simple $L$-module $V$ has a basis $(f_1, f_2)$, where $h \cdot f_1 = f_1$, $h \cdot f_2 = -f_2$, $y \cdot f_1 = f_5$, and $y \cdot f_2 = 0$. Now one puts

$$p = \sum_{t=0}^n (-1)^{n-t} \frac{n!}{t!} (y^t u') f_1 f_2^{n-t}$$

and one can verify directly that $h \cdot p = 0 = y \cdot p$, which implies that $p$ belongs to $P^L$. The homomorphism $\eta$ sends $f_1$ to 1 and $f_2$ to 0, so that $\eta(p) = y^n u' = u$.

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