Minimax Principles for the Solution of Semilinear Gradient Operator Equations in Hilbert Space

MELVYN S. BERGER

The Institute for Advanced Study, Princeton, New Jersey 08540

Communicated by Marston Morse, February 12, 1973

ABSTRACT A new variational characterization of solutions for an important class of nonlinear operator equations is obtained. The result obtained is used to derive sharp necessary and sufficient conditions for the solvability of such operator equations. Examples of the applicability of the results obtained to nonlinear Dirichlet problems and global differential geometry are discussed.

The importance of studying critical points that may not render an absolute minimum to a $C^1$ functional $\mathcal{I}(u)$ has long been emphasized by Marston Morse (1, 2). Here, I take up an important basic problem in which this point of view is essential. I seek a precise nonlinear analogue for the classic solvability theory of Fredholm (in the self-adjoint case). Thus (1) in the linear case, if $L$ is a bounded self-adjoint Fredholm operator mapping a Hilbert space $H$ into itself, the operator equation $Lu = f$ is solvable if, and only if, $f$ is orthogonal to the kernel of $L$.

On the other hand, a solution $u$ of $Lu = f$ (if it exists) is also a critical point of the functional $\mathcal{I}(u) = \frac{1}{2}(Lu, u) - (f, u)$ which generally has the following crucial properties:

(a) if the negative spectrum of $L$, $\sigma_-(L)$, is infinite, $u$ will have infinite Morse index;

(b) if $\sigma_-(L)$ is nonempty, $u$ will be neither an absolute nor even a relative minima for $\mathcal{I}$;

(c) if Ker $L$ is nontrivial, $u$ will be a degenerate critical point of $\mathcal{I}$ on $H$.

In the linear case, these difficulties can be circumvented by using the notions of orthogonality, orthogonal complement, and the spectral theory for self-adjoint operators. However, in the nonlinear case, these notions are not directly available, so that a nonlinear analogue of (1) must be pursued by alternative means. In this note, I show that a variational characterization of a tentative solution of the operator equation in question yields a precise analogue for (1) in the semilinear case. The result obtained substantially extends our earlier discussion of the problem (3). The full details of the results obtained will appear in a forthcoming paper.

VARIATIONAL PRINCIPLES FOR SEMILINEAR GRADIENT OPERATOR EQUATIONS

Let $H$ be a Hilbert space over the reals and $\Phi(H)$ be the set of bounded Fredholm linear operators mapping $H$ into itself. Then a nonlinear operator $g$, defined on $H$, is a completely continuous perturbation $C$ (not necessarily linear) of an element $L$ of $\Phi(H)$, i.e., $g = L + C$. Here we assume that (1) $C$ is a gradient operator so that there is a $C^1$ functional $\mathcal{I}(u)$ such that the Frechet derivative of $\mathcal{I}(u)$, $\mathcal{I}'(u) = Cu$ for each $u \in H$ and (ii) $L$ is self-adjoint. Thus, for $f \in H$, the solutions of $g(u) = f$ correspond precisely to the critical points of the $C^1$ functional

$$
\mathcal{I}(u) = \frac{1}{2}(Lu, u) + \mathcal{I}(u) - (f, u).
$$

Now we suppose that the equation $g(u) = f$ is solvable in $H$, and we seek a variational characterization of an associated critical value of $\mathcal{I}(u)$ defined by Eq. [1]. If $\mathcal{I}(u) \equiv 0$, such a characterization is easily obtained, as follows: Let $H_+$ and $H_-$ be the closed linear subspaces of $H$ on which the quadratic form $(Lu, u)$ is positive definite and negative definite, respectively. Let an arbitrary element $u$ of $H$ be written $u = x + y + z$, where $x \in H_+$, $y \in H_-$, and $z \in \text{Ker} L$. Then, it is easily proved that, if finite, the real number

$$
c = \inf_{H_+} \sup_{H_-} \inf_{\text{Ker} L} \mathcal{I}(x + y + z)
$$

is a critical value for $\mathcal{I}$. Furthermore, by (1), $c$ is finite if, and only if, the equation $Lu = f$ is solvable. Thus it is natural to attempt to extend the variational characterization [2], to a semilinear context with $\mathcal{I}(u) \not\equiv 0$, directly by studying the numbers

$$
c_1 = \inf_{H_+} \sup_{H_-} \inf_{\text{Ker} L} \mathcal{I}(x + y + z)
$$

$$
c_2 = \inf_{H_+} \sup_{H_-} \inf_{\text{Ker} L} \mathcal{I}(x + y + z).
$$

In fact, the following result holds.

THEOREM 1. Suppose the semilinear gradient operator $L + \mathcal{I}'$ satisfies the following conditions:

(i) for fixed $x, y, \mathcal{I}(x + y + z)$ is strictly convex (strictly concave) in $z$;

(ii) for $u \in \mathcal{I}' = \{u | \mathcal{I}'(u) - f \perp \text{Ker} L\}$, $\mathcal{I}(y) = \mathcal{I}(x + y + z)$ is strictly concave in $y$, and $\mathcal{I}(y) \to -\infty$ as $\|y\| \to \infty$;

(iii) for $u \in \mathcal{I}' = \{u | Lu + \mathcal{I}'(u) - f \perp H_-\} \cap \mathcal{I}'$, $\mathcal{I}(u) \to -\infty$ as $\|u\| \to \infty$.

Then the number $c_1$ (or $c_2$), if $\mathcal{I}(u) \neq \phi$, is a critical value for $\mathcal{I}(u)$ on $H$.

The proof is based on the

LEMMA. Suppose $c_1$ as defined above is finite. Suppose (i) for fixed $x, y$, the infimum over $\text{Ker} L$ is attained at $z = z(x, y)$, (ii) for fixed $x$, the supremum of $\mathcal{I}(x + y + z(x, y))$ over $H_-$ is attained at $y = y(x)$ with $z(x, y)$ and $y(x)$ differentiable functions of their arguments, and (iii) the infimum over $H_+$ in Eq. [3] is attained at $z$. Then $c_1$ is a critical value for $\mathcal{I}(u)$, and $\bar{u} = \bar{x} + y(\bar{x}) + z[\bar{x}, y(\bar{x})]$ is a critical point for $\mathcal{I}(u)$ with $\mathcal{I}(\bar{u}) = c_1$. An analogous result also holds for $c_2$. 

1519
Clearly, Theorem 1 can be considered as an extension of the von Neumann minimax theory of saddle points.

**Solvability of Semilinear Gradient Operator Equations**

We now determine necessary and sufficient conditions for the solvability of the operator equation

$$Lu + \nabla^2(u) = f$$  \[4\]

by using the extended minimax principles of Eq. [3].

**Theorem 2.** Under the hypothesis of Theorem 1, the operator Eq. [4] is solvable if, and only if, the set $\mathbb{M} = \{u|\nabla^2(u) - L \text{Ker } L\}$ is nonvacuous; moreover, if solvable, a critical value of Eq. [1] is given by one of the formulas in Eq. [3].

The proof is based on the observations that under the hypothesis of Theorem 1, $c_1$ (or $c_2$) are infinite if, and only if, the Eq. [4] is not solvable. On the other hand, the nonvacuousness of $\mathbb{M}$ is not only a necessary condition for the solvability of Eq. [4], but also a sufficient condition for the finiteness of $c_1$ (or $c_2$).

**Theorem 2** may be used to establish the openness of the range of mappings $L + \nabla^2$, and consequently (by results of Smale) to connect our results with the Morse theory of non-degenerate critical points on $H$.

**Some Examples**

I now consider some interesting examples from the theory of quasilinear elliptic partial differential equations that demonstrate the precision of Theorem 2.

**Example 1:** Consider the nonlinear Dirichlet problem

$$\Delta u + g(u) = f \quad u|_{\partial \Omega} = 0$$  \[5\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\Delta$ is the Laplace operator defined on $\Omega$ whose eigenvalues are denoted $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$. Assuming $f \in C^0(\Omega)$, $g \in C^2$ strictly convex function with $g(0) = 0$ and $\lim_{u \to \pm \infty} g'(u) = \beta(\pm \infty)$, $\lambda_1 < \beta(\pm \infty) < \lambda_2$, $0 < \beta(- \infty) < \lambda_1$, Prodi and Ambrosetti found the following sharp results (unpublished). The image of the singular points of the mapping $\Delta u + g(u) : C^0(\Omega) \to C^0(\Omega)$ is a connected $C'$ manifold $\mathbb{M}$ of codimension 1 in $C^0(\Omega)$ dividing $C^0(\Omega)$ into two components $U$ and $V$. For $f \in U$, Eq. [5] has exactly two solutions, $f \in V$, Eq. [5] has no solutions, while for $f \in \mathbb{M}$, Eq. [5] has exactly one solution. Since the solutions of Eq. [5] are precisely the critical points of the functional

$$J(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - G(u) - f u\right) \, dV$$

in the Sobolev space $W_{1,2}(\Omega)$, the Equation [5] can be written in the form of [4]. Moreover, one easily sees that, for $f \in \mathbb{M}$, the unique solution $u(f)$ is a degenerate critical point of the functional Eq. [6], while for $f \not\in U$, the corresponding two solutions $u_1(f), u_2(f)$ are nondegenerate critical points of Morse index 0 and 1, respectively. Setting $H = W_{1,2}(\Omega)$ and letting $L$ denote the abstract operator representing $(\Delta + \lambda_1)$, we observe that, in this case, $H_+ = \phi$, so the critical points just mentioned cannot, however, be described by the formulae [3]. However, a variant of Theorem 2 holds, giving a precise criterion for $f \in U \subset \mathbb{M}$.

**Example 2 (4,5): Conformal Metrics of Prescribed Gauss Curvature $K(x)$ on the Sphere $S^2.$** The problem consists in finding a $C^\infty$ function $u$ defined on $S^2$ (with the standard metric $g_1$) such that $\bar g = e^{2u}g$ is a metric of $C^\infty$ Gauss curvature $K(x),$ given sup $\sqrt{g_1}K(x) > 0$. Equivalently, we solve the quasilinear elliptic equation on $(S^2, g_1)$

$$\Delta u - 1 + Ke^{2u} = 0$$

The solutions of Eq. [7] correspond exactly to the critical point of the functional restricted to the Sobolev space $W_{1,2}$ $(S^2, g_1)$

$$J(u) = \int_{S^2} \left\{ \frac{1}{2} |\nabla u|^2 + u - \frac{1}{2} Ke^{2u} \right\} \, dV.$$  \[8\]

By setting the Laplace Beltrami operator $\Delta = L$ and observing that $H_+ = \phi$ and Ker $L$ consists of the constants, we observe that Eq. [7] is solvable for any $K(x)$ admissible, provided the hypotheses of Theorem 1 and in particular (iii) are satisfied. Moser (unpublished) proved a difficult estimate demonstrating that (iii) holds if $K(x) = K(-x),$ but not necessarily in general. In ref. 5, Kazdan and Warner found an admissible $C^\infty$ function $K(x)$ for which Eq. [7] is not solvable. Thus, some condition such as (iii) is certainly necessary for the validity of Theorem 2. Moreover, if $K(x) = K(x)$ Theorem 2 implies [7] is solvable if, and only if, sup $\sqrt{g_1}K(x) > 0$ (solving the geometric problem in this case).

This research was partially supported by National Science Foundation Grant GP-36418X and AFOSR Grant 79-2347.