On completeness of the Bergman metric and its subordinate metric

(Carathéodory differential metric/Weil-Petersson metric/Teichmüller space)

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ABSTRACT

It is proved that on any bounded domain in the complex Euclidean space \( C^n \) the Bergman metric is always greater than or equal to the Carathéodory distance. This leads to a number of interesting consequences. Here two such consequences are given. (i) The Bergman metric is complete whenever the Carathéodory distance is complete on a bounded domain. (ii) The Weil-Petersson metric is not uniformly equivalent to the Bergman metric in the Teichmüller space \( T(g) \) of any Riemann surface of genus \( g \geq 2 \).

Let \( D \) be any bounded domain in the complex Euclidean space \( C^n (n \geq 1) \). Then \( D \) admits both the Bergman metric \( s_D \) and the Carathéodory differential metric \( a_D \). The main purpose of this note is to prove the following:

**THEOREM 1.** Let \( D \) be a bounded domain in \( C^n \). For each \( z \in D \) and \( \xi \in C^n \)

\[
\alpha_D(z,\xi) \leq s_D(z,\xi).
\]

Let \( \rho_D \) and \( d_D \) be the distance functions on \( D \) which are induced from \( \alpha_D \) and \( s_D \), respectively. Then the Carathéodory distance \( c_D \) satisfies:

\[
\rho_D \leq \rho_D \leq d_D.
\]

From this observation, we have

**THEOREM 2.** Let \( D \) be any bounded domain \( C^n \). Then the Bergman metric \( s_D \) is complete in \( D \) if the Carathéodory distance \( c_D \) is complete.

Eichler, S. Wolpert and ref. 1 and T. Chu have independently proved that the Weil-Petersson metric is not complete in the Teichmüller space \( T(g) \) of a compact Riemann surface of genus \( g \geq 2 \). On the other hand, it follows from Theorem 2 and the completeness of the Carathéodory distance in \( T(g) \) (see ref. 2) that the Bergman metric is complete in \( T(g) \). Therefore, we conclude the following:

**THEOREM 3.** In the Teichmüller space \( T(g) \) of any Riemann surface of genus \( g \geq 2 \), the Weil-Petersson metric is not uniformly equivalent to the Bergman metric.

For the proof of Theorem 1 we need the following

**LEMMA.** Let \( f \) be a holomorphic function on \( D \) satisfying \( |f(z)| \leq M, M > 0 \). Then for each \( z \in D \) and each \( \xi \in C^n \)

\[
\left| \sum_{\alpha=1}^{n} \xi_{\alpha} \frac{\partial f}{\partial \xi_{\alpha}} \right| \leq M^2 \sum_{\alpha=1}^{n} \left| \xi_{\alpha} \right| \log K_D(z,\xi),
\]

where \( K_D \) is the Bergman kernel function of \( D \).

**PROOF:** Set \( a(t) = f(t)K_z(t), b(t) = \sum_{\alpha=1}^{n} \xi_{\alpha}(\partial / \partial z_{\alpha})[K_z(t)/K_z(z)], \) and \( (a, b) = \int_D \hat{a} \hat{b} \omega \), where \( \omega \) is the Lebesgue measure in \( D \) and \( K_z(t) = K_D(t, z) \).

By the Schwarz inequality

\[
(a, b) \leq (a, a)_{\alpha}(b, b)_{\alpha}.
\]

Using the reproducing property of \( K_D \) (ref. 3), we have

\[
(a, a) = \iint K_\alpha K_\alpha = M^2 K_D(z, \bar{z}), \quad (b, b) = \sum \xi_{\alpha} \xi_{\alpha} = \sum \xi_{\alpha} \xi_{\alpha}.
\]

Therefore, we prove that \( (a, b) \)

\[
= \sum \xi_{\alpha} \xi_{\alpha}(\partial / \partial z_{\alpha})[K_\alpha K_\alpha] = \sum \xi_{\alpha} \xi_{\alpha} \log K_D(z, \bar{z})\]

From [1] together with [2], [3], and [4], Lemma follows.

**Theorem 1** is now an immediate consequence of Lemma when we recall that the Carathéodory differential metric of \( D \) is given (see ref. 4) by

\[
\alpha_D(z,\xi) = \sup \left\{ \int_{\partial D} \frac{\partial f}{\partial \xi_{\alpha}} \xi_{\alpha} : f \in H(D, \Delta) \right\},
\]

where \( H(D, \Delta) \) denotes the class of holomorphic functions \( f \) on \( D \) with values in the unit disc \( \Delta \) in \( C \).

It should be noted that the method of the proof of the above lemma was essentially due to K. H. Look in ref. 5.

The author acknowledges that the above simplification of the proof of Theorem 1 was first suggested by J. Burbea.