A reducibility criterion for generalized principal series
(semisimple Lie groups/representations of groups)

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ABSTRACT We give necessary and sufficient conditions for the reducibility of representations of semisimple Lie groups induced from discrete series representations of parabolic subgroups, when the inducing parameters are nonsingular.

Let G be a connected semisimple Lie group with finite center and P ⊂ G, a cuspidal parabolic subgroup. Fix a Cartan involution θ of G, and let P = MAN be the associated Langlands decomposition of P. Thus, M is a reductive group containing a compact Cartan subgroup T+, A is a vector group, N is the unipotent radical of P, and H = T+A is a Cartan subgroup of G. We choose T+ so that H is θ-stable. For simplicity, we assume in what follows that G is linear and acceptable in the sense of Harish-Chandra (1); these assumptions are unimportant for the results, but they simplify the notation slightly.

The Killing form of G and its various restrictions and dualizations are written , . Lie algebras are denoted by small German letters; thus, g is the Lie algebra of G, etc. Fix a (nonunitary) character γ: H → C; then γ = (λ, ρ) = λ ⊗ ρ, where λ is a character of T+ and ρ is a character of A. We denote the differentials of these characters by the same letters. We assume that λ is nonsingular with respect to M—i.e., that for every root α of ξC, (α, λ) ≠ 0. In this case, Harish-Chandra (ref. 1, pp. 176–178) has constructed a certain representation δα in the discrete series of M. Then, δα ⊗ ρ ⊗ 1 is a representation of P = MAN; by (normalized) induction we obtain a representation πγ = IndP (δα ⊗ ρ ⊗ 1) of G, which we call a generalized principal series representation.

THEOREM With notations as above, let πγ be a generalized principal series representation of G. Assume that γ is nonsingular—i.e., that for every root α of A in gC, (α, γ) ≠ 0. Then πγ is reducible if and only if there exists a root α of A in gC such that:

- (i) α is a real root, and the following parity condition holds: let φα: SL(2, R) → G be defined in the obvious way using α, and let mα = φα(2I) ∈ T+. Then n should be odd or even according to whether γ(mα) is 1 or −1.

Many special cases of this theorem have been proved previously (refs. 2, 3, and the references therein; T. Enright, personal communication).

The major new technique involved in the proof of this Theorem is a careful examination of the notion of “coherent continuation” of characters [defined by Schmid (6)], which amounts to a study of tensor products of finite and infinite dimensional representations. Coherent continuation within a fixed Weyl chamber has been treated thoroughly by Zuckerman (personal communication), but we consider also continuation across walls into adjacent chambers. In addition, Langlands’ results (4) [as exploited by Speh (5)] and the ideas of Vogan (unpublished) are used as reduction techniques. We also use the character identities of Schmid (6).

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