On unstable cohomology classes of $SL_n(Z)$

(stable cohomology/special linear group/Riemannian symmetric space/Borel-Serre compactification/algebraic K-theory)

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ABSTRACT From algebraic K-theory, we show that there exists a spectral sequence that has real cohomology of $SL_n(Z)$ as its $E_1$-terms and converges to the tensor product of a polynomial algebra and an exterior algebra. On the basis of this spectral sequence, we discovered several families of real unstable cohomology classes of $SL_n(Z)$.

Let $SL_n(Z)$ denote the special linear group of $n$-by-$n$ matrices and let $SL(Z) = \lim SL_n(Z)$ be the infinite special linear group. In ref. 1, A. Borel studied the real stable cohomology $H^*(SL(Z))$ and showed that within a certain stable range, the real cohomology $H^*(SL_n(Z))$ of $SL_n(Z)$ is an exterior algebra with generators $h_i$ at dimension $2i - 1$ ($i = 3, 5, \ldots$). This note is a preliminary announcement of some families of cohomology classes discovered recently lying outside of the stable range. The constructions of these classes are closely related to those of the stable classes $h_i$, and for this reason we call them the secondary classes.

Statement of the main results

THEOREM 1. (Stable range condition.) Let $\bar{h}_i$ denote the pull back of the stable class $h_i$ to the cohomology $H^{2i-1}(SL_n(Z))$. Then for every $i \geq n$ the cohomology class $\bar{h}_i = 0$, and for all $3 \leq i < n - 1$ all the classes within this range are nonzero and generate an exterior subalgebra in $H^*(SL_n(Z))$.

For a finite sequence of odd integers $I = [i_1, i_2, \ldots, i_k]$ all bigger than 1, we use the symbol $\bar{h}_I$ to denote the element $\bar{h}_{i_1}\bar{h}_{i_2} \ldots \bar{h}_{i_k}$ in the exterior algebra $H^*(SL_n(Z))$. By the height of $\bar{h}_I$ we mean the highest of all these integers $i_1, \ldots, i_k$, and we use the symbol

$$ht(\bar{h}_I) = \max (i_1, \ldots, i_k).$$

Note that the dimension of $\bar{h}_I$ is given by the formula

$$\dim \bar{h}_I = \dim \bar{h}_{i_1} + \dim \bar{h}_{i_2} + \ldots + \dim \bar{h}_{i_k}.$$

THEOREM 2. (Secondary cohomology classes.) Let $X$ be the bounded symmetric domain $SL_n(R)/SO_n(R)$ and let $Y$ be the quotient space $SL_n(Z)/X$ under the action of $SL_n(Z)$. Then

(a) For every sequence of elements $\bar{h}_{i_1}, \ldots, \bar{h}_{i_k}$ satisfying the condition

$$ht(\bar{h}_{i_1}) + ht(\bar{h}_{i_2}) + \ldots + ht(\bar{h}_{i_k}) = n,$$

there is a well-defined cohomology class with compact support $[\bar{h}_{i_1}] [\bar{h}_{i_2}] \ldots [\bar{h}_{i_k}]$ in $H^d(Y)$ whose dimension $d$ is given by the formula

$$d = \dim \bar{h}_{i_1} + \dim \bar{h}_{i_2} + \ldots + \dim \bar{h}_{i_k} + (k - 1).$$

(b) For every sequence of elements $\bar{h}_{i_1}, \ldots, \bar{h}_{i_k}$ satisfying the condition

$$ht(\bar{h}_{i_1}) + ht(\bar{h}_{i_2}) + \ldots + ht(\bar{h}_{i_k}) = n - 1,$$

there is a well-defined cohomology class with compact support $[\bar{h}_{i_1}] [\bar{h}_{i_2}] \ldots [\bar{h}_{i_k}]$ in $H^d(Y)$ whose dimension $d$ is given by the formula

$$d = \dim \bar{h}_{i_1} + \dim \bar{h}_{i_2} + \ldots + \dim \bar{h}_{i_k} + k.$$

(c) If we permute the order of the elements $\bar{h}_{i_1}, \ldots, \bar{h}_{i_k}$, then the corresponding classes differ only up to sign. Except for these permutation relations, the above cohomology classes in (a) and (b) form a set of linearly independent elements in $H^d(Y)$.

Note that $Y_n$ is an orientable real homology manifold and so by Poincare duality, the cohomology with compact support $H^d(Y_n)$ is the same as the real homology $H_{n-d}(Y_n)$ in which $m = 1$. In this way, the theorem above gives information about the existence of nonzero secondary classes in $H^*(SL_n(Z))$.

Motivation from algebraic K-theory

It is well known from algebraic K-theory that the stable cohomology $H^*(SL)$ is related to the cohomology of the loop space of $BO$ because of the homotopy equivalence between $X \times BSL^+$ and the loop space of $BO$. In fact, Borel's result for $SL(Z)$ implies that the cohomology of $BO$ is the tensor product of an exterior algebra $E(u)$ with one generator $u$ of degree 1, and a polynomial algebra with generators $x_i$ given by the transgression of $h_i$. On the other hand, there is a natural filtration of $BO$ by subspaces

$$BO_1 \subset BO_2 \subset \cdots \subset BO_n \subset \cdots$$

(see ref. 2) so that the cohomology of $BO$ can be computed from a spectral sequence whose $E_1$-term is given by

$$E_1^{p,q} \cong H^p + q(BO_p,BO_{p-1}).$$

In addition, this relative cohomology $H^p(BO_n,BO_{n-1})$ can be identified with the cohomology $H^p(Y_n)$ of $Y_n$ with compact support, and as before this cohomology of compact support is related to the real cohomology $H^*(SL_n)$ by Poincare duality. In this manner the knowledge of the structure of $H^*(BO)$ already predicts that nontrivial cohomology classes exist. Our results in Theorems 1 and 2 arise from this motivation, and they provide all the necessary ingredients for the spectral sequence to converge to the algebra $H^*(BO)$.

Outline of the proof of Theorem 1

Because the proofs of Theorems 1 and 2 require careful geometric construction, it is planned to present the detailed proof later, and we will indicate here some of the ideas involved in the case when $n$ is odd. First, the cohomology $\bar{h}_I$ is defined by

$${}^1$$ Because $SL_n(Z)$ has torsion, it does not operate freely on $X_n$ and so the space $Y_n$ looks locally like the quotient space of a Euclidean space by finite group, i.e., V-manifold (in the sense of Satake).
means of an invariant differential form $Z_i$ on the bounded symmetric domain $X = \text{SL}_n / \text{SO}_n$, which, in turn, is defined by taking the left translation of an alternating $(2t - 1)$-form $Z_i$ on the space $\text{sl}_n / \text{so}_n$. From this definition, it is easy to see that $\delta_i$ must vanish even on the Lie algebraic level for $i > n$. To obtain the precise information for $i = n$ and the rest of Theorem 1, we need the Borel–Serre compactification $\tilde{Y}_n$ of $Y_n$, and a “nice” embedding $f : \tilde{Y}_n \to Y_n$. Now the Borel–Serre boundary $\partial Y_n = Y_n - \tilde{Y}_n$ is obtained from gluing together manifolds with corners $W_i$, $1 \leq i \leq n - 1$, each of which corresponds to a maximal parabolic subgroup. The embedding $f$ is chosen in such a manner that the pull back of the differential form $Z_n$ to each boundary component $W_i$ is the zero form. It follows that if we give a suitable smooth structure on the manifold with corners $\tilde{Y}_n$, the differential form $f^* Z_n$ defines a cohomology class with compact support $[\tilde{\gamma}_n]$ in $H^*_c (\tilde{Y}_n)$. Note that we have defined here the cohomology class $[\gamma_n]$ in Theorem 2. Moreover, the cup product of $[\gamma_n]$ with the cohomology class $\gamma_1 \cdots \gamma_{n-2}$ is the fundamental class in $H^*_c (\tilde{Y}_n)$. This implies that in the cohomology $H^*_c (\tilde{Y}_n)$ all the cohomology classes $\gamma_i$ with $ht(\gamma_i) < n - 1$ are nonzero as in Theorem 1, and in the cohomology with compact support $H^*_c (\tilde{Y}_n)$ all the classes $[\gamma_i]$ with $ht(\gamma_i) = n$ are nonzero as in Theorem 2.

**Definition of the cohomology class $[\gamma_1] \cdots [\gamma_k]$**

For $k = 1$, the cohomology class $[\gamma_i]$ with $ht(\gamma_i) = n$ is defined by observing that the differential from $Z_i$ vanishes along the boundary components as before. We now describe the definition of the class $[\gamma_1] \cdots [\gamma_k]$. Let $n_1 = ht(\gamma_1)$ and $n_2 = ht(\gamma_2)$. Then $n = n_1 + n_2$, and there is a maximal parabolic group defined by the matrices $(X_1)$ with $X_1 = 0$ for $i < n_1$, $j > n_1$. If $W$ is the boundary component corresponding to this maximal parabolic group, then $W$ is the total space of a torus bundle over the manifold $\tilde{Y}_n \times \tilde{Y}_n$. This gives the following composite homomorphism:

$$H^k_c (\tilde{Y}_n \times \tilde{Y}_n) \to H^k_c (W) \to H^k (\partial \tilde{Y}_n) \to H^{k+1}_c (\tilde{Y}_n).$$

The cohomology class $[\gamma_1] [\gamma_2]$ is defined to be the image under this mapping of the element $\gamma_1 \otimes \gamma_2$ in $H^2_c (\tilde{Y}_n) \otimes H^2_c (\tilde{Y}_n)$, and so its dimension is given by the formula:

$$\dim [\gamma_1] [\gamma_2] = \dim \gamma_1 \otimes \gamma_2 + 1 = \dim \gamma_1 + \dim \gamma_2 + 1.$$

We omit the definition of the general case because it is obtained by repeating the same procedure.

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