One class of meromorphic solutions of general two-dimensional nonlinear equations, connected with the algebraic inverse scattering method

(nonlinear partial differential equations/complete integrability)

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ABSTRACT For systems of nonlinear equations having the form \[ [L_n - (\partial/\partial y)] L_m - (\partial/\partial y)] = 0 \]
the class of meromorphic solutions obtained from the linear equations
\[ \phi_i = e^{L_n \alpha_i}, \quad \phi_f = e^{L_m \alpha_f} \]
is presented.

The most general form of "exactly solvable" (or "completely integrable") systems of nonlinear two-space dimensional (in \( x \) and \( y \)) partial differential equations can be written as a condition of commuting of two differential operators \((1, 2)\). Non-trivial two-dimensional systems in variables \( x, y, \) and \( t \) can be obtained if each of these operators have differentiation, say in \( x, \) and differentiation of order only one in \( y \) and \( t \) \((3)\). Such systems were at first introduced by Zakharov and Shabat \((1)\) as a natural generalization of Lax pair \((4)\). These systems are the culmination of investigations started by Burchannal and Chaundy \((5)\) and we write them in the form
\[
\frac{dL_m}{dt} - \frac{dL_n}{dy} = [L_n, L_m],
\]
where \( L_n \) and \( L_m \) are differential operators \( L_n = \sum_{i=0}^{n} u_i (d/dx)^i \) and \( L_m = \sum_{i=0}^{m} v_i (d/dx)^i \), of orders \( n \) and \( m \), respectively, in which \( u_0 = v_0 = 1 \), \( u_{n+1} = v_{m+1} = 0 \).

I present here a special class of solutions of this equation based on the following two observations. First of all, we showed previously \((6, 7)\) that for solutions \( u(x, y, t) \) which are meromorphic in \( x \) the evolution of poles \( a_i(y, t) \) is described as an evolution in \( y \) and \( t \) directions according to two Hamiltonian flows commuting simultaneously with the flow describing the interaction of particles with an inverse square potential. On the other hand it was shown in \((6, *)\) that there are linearized equations, the so called higher Burgers–Hopf (BH) equations, the poles of meromorphic solutions of which evolve also according to these Hamiltonian flows. These two observations lead to the conclusion that it is possible to construct directly the solutions of two-dimensional systems starting from linear partial differential equations. In this paper, I prove this and present a method for the construction of this class of solutions. It is possible to construct them starting from the Gelfand–Levitan–Marchenko equation. Instead of doing this, I derive meromorphic solutions directly, by commutativity conditions. I also examine the behavior of their poles.

The evolution of poles of most nonlinear, completely integrable, systems is connected with the Hamiltonian for a system of particles interacting via the potential \( GP(x) \) where \( P(x) \) is the Weierstrass elliptic function.

Thus, we consider the Hamiltonian \((7)\)
\[
H_p = \frac{1}{2} \sum_{i \in I} b_i^2 + G \sum_{i < j} P(a_i - a_j)
\]
for an arbitrary number of particles \( a_i; i \in I \). This Hamiltonian has infinitely many first integrals. These integrals come from the Lax representation
\[
dL/dt = [A, L]
\]
for system \((1)\), where the matrix \( L = (L_{ij})_{i \in I} \) in \((2)\) has the form
\[
L_{ij} = (1 - \delta_{ij}) \sqrt{-G} (a_1 - a_j) + \delta_{ij} b_i,
\]
with \( \alpha^2(x) = \mathcal{P}(x) \). Then the functionals \( J_n = 1/n \text{tr}(L^n), n \geq 1 \) are the first integrals of \( H_p \). Moreover, \( J_n \) are in involution and they are sums of polynomials in \( d_i, \mathcal{P}(a_i - a_j), \) and \( G \) with rational coefficients. The form of the first terms of \( J_n \) is the following
\[
J_n = \frac{1}{n} \sum_{i \in I} b_i^n + G \sum_{i < j} (b_i^{n-2} + b_j^{n-2} + \ldots + b_j^{n-2}) \mathcal{P}(a_i - a_j) + \ldots
\]

For the degenerate case \( \mathcal{P}(x) = x^{-2} \) and finite \( I \) there are very simple formulae \((6, 7)\) for the solution of the Cauchy problem for any \( J_n \).

If we consider such trajectories \( x(t) \) of Hamiltonian \( J_n \) that all the integrals \( J_m \) vanish, \( J_m = 0, m = 1, 2, \ldots \), then we come to the poles of higher BH equations.

As is well known \((8)\), the usual BH equation has the form \( u_t = u_{xx} - 2uu_x \). The higher BH equations, like the ordinary one, are linearized by the Hopf–Cole substitution \( u = -(\log \phi)_x \). Thus, \( nth \) higher BH equation has the form \( u_t = -d/d\xi (\partial^n/\partial \xi^n \exp[-\int u dx]) \exp[\int u dx] = \mathcal{H}_n[u], \) or \( \mathcal{H}_n[u] \) can be defined as a polynomial in \( u, u_x, \) and \( u_{xx} \) by induction
\[
\mathcal{H}_n[u] = \frac{d}{dx} C_n[u]
\]
and \( C_{n+1}[u] = d/dx C_n[u] - u C_n[u], C_0[u] = -1 \).

I must emphasize that the notations here and in ref. 6 differ only in the sign of \( u \). Then, the basic information on higher BH is contained in the following

Abbreviation: BH equation, Burgers–Hopf equation.


Herman-Chudnovsky, January 29, parts 1 and 2.
THEOREM 1. The equation $u_t = BH_u[u]$ reduces by the transformation $u = - \log \varphi$, to the equation $\varphi_t = \varphi_x + \varphi_y$. Meromorphic solutions $u(x, t)$ of $u_t = BH_u[u]$ have the form $u(x, t) = - \sum_{i=1}^n (x - a_i)^{-1}$ for $u_i = a_i(t)$; $i \in I$ and meromorphic $u(x, t) = - \sum_{i \in I} (x - a_i)^{-1}$ satisfy $u_t = BH_u[u]$ if
\[\begin{align*}
-\dot{a}_i &= n! \sum_{|i|=j} (a_i - a_j)^{-1} \cdots (a_i - a_{m-1})^{-1}, i \in I.
\end{align*}\]
[5]

Also, the system [5] has a very simple description:

PROPOSITION 2. The system [5] is embedded into a system with Hamiltonian

\[J_n \text{ for } P(x) = x^{-2} \text{ and } G = -1.\]

Moreover, the system [5] corresponds to the following submanifold

\[b_t = \sum_{j \neq i} (a_i - a_j)^{-1}; i \in I.\]

For finite $I$, the trajectories $a_i(t); i \in I$ of $J_n$ with $G = -1$ on which all the integrals $J_m$ vanish, $J_m = 0$, are precisely the solutions of the system [5].

For any integer $n \geq 2, m \geq 2$ we consider the following system of equation in partial derivatives for functions $u_j(x, y, t)$ and $v_j(x, y, t), i = 1, \ldots, n, j = 1, \ldots, m$,

\[\begin{align*}
\frac{da_{ij}}{dt} &= \frac{da_{ij}}{dy} = [L_n, L_m],
\end{align*}\]

where

\[\begin{align*}
L_n &= \sum_{i=0}^{m} u_i, \quad L_m = \sum_{j=0}^{m} v_j, \quad u_t = v_t = 1, \quad [6]
\end{align*}\]

and other $u_i, v_j; i = 1, \ldots, n, j = 1, \ldots, m$ are defined by induction

\[\begin{align*}
u_k &= - \sum_{k+1}^{m} C_k^t w^{(k-1)} u_{k+1} - C_k^{m-1} w^{(m-k-1)},
\end{align*}\]

and $k = m - 2, \ldots, 0$.

\[\begin{align*}
u_k &= - \sum_{k+1}^{m} C_k^t w^{(k-1)} v_{k+1} - C_k^{m-1} w^{(m-k-1)},
\end{align*}\]

and $k = m - 2, \ldots, 0$.

Here we denote by $w^{(j)}$ the $j$th derivative of $w$; $w^{(0)} = w_t$, and $C_k^t$ are binomial coefficients.

Proof: As usual (9), we consider the common eigenfunction $\psi(x, y, t, k)$ for two operators $L_n - (\partial / \partial t)$ and $L_m - (\partial / \partial y)$, having the form

\[\begin{align*}
\psi(x, y, t, k) &= (1 + wk^{-1}) \exp(kx + k^*t + km^y),
\end{align*}\]

where $k$ is a spectral parameter. Then, it is clear that the function $\psi$ from [11] is the eigenfunction for $L_n - (\partial / \partial t)$ and $L_m - (\partial / \partial y)$:

\[\begin{align*}
L_n \psi &= \psi_t, \quad L_m \psi &= \psi_y.
\end{align*}\]

Iff the following system of equations in $w, u_i$, and $v_j$ is satisfied:

\[\begin{align*}
u_m &= - \sum_{s=1}^{n^2} C_s^{k+1} w^{(s-k-1)} u_s - C_s^{m-1} w^{(m-k-1)}, m = n - 2, \ldots, 0 \quad [13]
\end{align*}\]

\[\begin{align*}
u_i &= \sum_{s=0}^{n^2} w^{(s)} u_s + w^{(n)} \quad \text{and an an}
\end{align*}\]

analogous system for $v_j$ changing $u_t$ to $v_t$ and $n$ to $m$. Then $v$ depends on an arbitrary parameter $k$ and $\psi(k)$ is a null function for the commutator $[L_n - (\partial / \partial t), L_m - (\partial / \partial y)]$. This operator is the operator on $\partial / \partial x$ and so cannot have infinite dimensional null subspace unless it is zero. Thus, [6] is satisfied when [13]–[14] for $w, u_t$, and $v_j$ is true. By induction, it is easy to show that, assuming [13], the right side in [14] is exactly $BH_n[w]$. Because [9] is equivalent to [13] and [8] to [14], the theorem is proved.

COROLLARY 4. For any $h(x)$ and any solution $\varphi(x, y, t)$ of the linear problem

\[\begin{align*}
\varphi_t &= \varphi_x + \varphi_y = \varphi_{x-x}
\end{align*}\]

with

\[\begin{align*}
d^2 \log \varphi(x, 0, 0) = h(x)
\end{align*}\]

there exists a solution $u, v_j$ of [6] defined by [9] and [10], where

\[\begin{align*}
w(x, y, t) &= \frac{-d}{dx} \log \varphi(x, y, t).
\end{align*}\]

Exact formulas for solutions [9], [10], and [14] can be obtained from [15].

On the other hand, formulas for rational solutions of [6] having the form [8]–[10] can be written, using Theorem 1 and Proposition 2 and results from refs. 10 and 11.

In fact, if initial conditions $a^{i_0}, \ldots, a^{i_n}, a^{j_0}, a^{j_m}, a^{i_0}, a^{i_0}$ satisfy

\[\begin{align*}
a^{i_0} &= n! \sum_{|i| \neq |j|} (a^{i_0} - a^{i_1})^{-1} \cdots (a^{i_0} - a^{i_m})^{-1};
\end{align*}\]

\[\begin{align*}
a^{j_0} &= m! \sum_{|j| \neq |i|} (a^{j_0} - a^{j_1})^{-1} \cdots (a^{j_0} - a^{j_m})^{-1};
\end{align*}\]

and

Then the function $w(x, y, t) = \Sigma t(x - a_t)^{-1}$, being the solution of [8], can be written as

\[\begin{align*}
w(x, y, t) &= - \frac{d}{dx} \log \chi(x, y, t).
\end{align*}\]

Here $\chi(x, y, t) = \chi_k(x)$ is a characteristic polynomial on $x$ of matrix $M(y, t)$: $M(y, t) = \text{diag}(a^0_1, \ldots, a^0_n) + (y - y_0) L^m(a^0_0, a^0_m) + (t - t_0) L^n(a^0_0, a^0_n)$, where $L(a, b)$ is the matrix for $P(x) = x^{-2}, G = -1$.

The solution [17] $w(x, y, t) = - \Sigma t(x - a_t)^{-1}$ satisfies [8] provided

\[\begin{align*}
a^{i_0}(t_0, y_0) = a^{i_0}_0, \quad a^{i_0}(t_0, y_0) = a^{i_0}_0, \quad a^{j_0}(t_0, y_0) = a^{j_0}_0.
\end{align*}\]

Results analogous to these can also be obtained for matrix differential operators.
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