Growth of complex systems can be related to the properties of their underlying determinants

(organismal biology/population biology/social systems/economics/organization)

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Communicated by Stanley M. Garn, August 22, 1979

ABSTRACT Growth—increase in size, number, or amount—in many cases appears to follow simple empirical laws. Such laws have been noted in a wide variety of fields for many years. Until now these laws have never been related to the underlying determinants of these systems. By starting with fundamental properties of the component mechanisms in such systems, one can derive a basic growth equation for which the well-known laws of growth are special cases.

Growth and morphogenesis are ubiquitous, particularly among more complex forms of organization such as biological organisms, populations, social and economic systems, and the products of human technology (1). Awareness of and interest in these two manifestations of complex organization probably go back to the prehistoric invention of counting. Early hunters and gatherers undoubtedly observed changes in time and space of the populations of animals and plants upon which their existence critically depended. Data pertaining to growth can be found in the earliest recorded history. Exponential growth was recorded in tablets of baked clay about 4000 years ago by the early Babylonians (3).

Inevitably, the accumulation of data led to the search for simple "laws of growth"—mathematical formulas that would provide a useful summary of large amounts of data and, by extrapolation, predictions concerning future growth. Often noted successes in this endeavor are the laws attributed to Malthus (4), concerning the unrestricted growth of populations, and to Pareto (5), dealing with the relative distribution of income among economic classes. More recently, in the field of biology, the work of Thompson (6) on growth and form and of Huxley (7) on relative growth have become classic. However, these empirical laws of growth and development have suffered relative neglect during the past several decades because they could not be related to the basic or fundamental underlying mechanisms that have become the focus of analytical science during this period.

The analytical study of growth has revealed in all cases a multitude of individual mechanisms that underlie what often appears to be a simple pattern of growth. In some cases, although nearly all the individual mechanisms are accounted for, still there is no understanding of how the overall pattern of growth results from the interaction of all these mechanisms. This is not too surprising because all the interesting examples of growth and development are found among complex systems. For this reason, it may be some time yet before the overall growth of a system can be related in detail to the properties of its underlying component mechanisms.

I have found that important qualitative aspects of growth can be derived from the properties of the underlying determinants. The formalism for accomplishing this is mathematical systems analysis. In this paper I shall (i) give an appropriate mathematical description of the component mechanisms, (ii) show how this information is combined to give a description of an integrated system, and (iii) present a basic growth equation that includes as special cases all the well-known growth equations. Elsewhere (8) I will show that the allometric law of relative growth also follows naturally according to this formalism and I will discuss the implications of these results for studies of growth and development. In principle, this formalism should permit a detailed, quantitative understanding of growth and development in terms of the underlying determinants.

Description of component mechanisms

There are often several ways to describe a given mechanism. When complex systems are considered, the preferred description is a mathematical one that is based on the essential features of the mechanism and yet is simple enough to be tractable when large numbers of mechanisms are considered in an integrated system.

Linear analysis is the most common description of this sort. There are two characteristic features of this formalism (9): (i) a change in an input variable of the mechanism produces a proportionate change in the output variables and (ii) the response to a sum of inputs is the sum of the responses to the individual inputs (i.e., the inputs act independently). Many mechanisms in nature are described by linear mathematics, at least over a range of values for the variables that is of practical interest.\(^a\)

Mathematical descriptions for mechanisms that are not adequately described by linear analysis have yet to be fully developed. For a large class of mechanisms that I shall call synergistic, the changes in the output variables are not proportional to the changes in the input variables; for example, they become less than proportional at high values of the input variables when saturation or diminishing return sets in. The response to a sum of inputs is not the sum of the responses to the individual inputs, but is a more complex relationship involving products of the individual input variables. In other words, the inputs no longer act independently, but synergistically. These types of behavior are characteristic of most complex systems. Examples of synergistic mechanisms include the association of substrate and enzyme to produce an activated intermediate capable of conversion to product\(^c\) (12, 13), the physical association of two

\(^a\) The earliest physical evidence of counting is a "tally stick," found in Czechoslovakia, that dates back 30,000 years to the Paleolithic period (2).

\(^b\) Familiar examples include the motion of a rigid body acted upon by several outside forces (10) and the electrical current produced in one branch of a complex network containing several voltage sources (11).

\(^c\) Antigen–antibody complexes, binding proteins for transport, hemoglobin, etc., may be considered as special cases that are very similar to the enzyme–substrate class.
species when a predator captures its prey (14–16), and the association of different factors of production to yield a product in an economic enterprise (17, 18). In contrast to the examples in the previous paragraph, where the individual inputs are essentially of the same kind and each simply makes its independent contribution to the output along with that of the others, the inputs in the examples given above are distinct and must associate with others to produce an output.

The mathematical description for the rate of an elemental association involves several probability considerations, such as the probability of two elements being in the same place at the same time and of their having the appropriate disposition, energy, etc. to form a relatively stable complex. This description for the rate of association is most conveniently represented as a phenomenological relation involving the product of the concentrations or amounts of the two elements and a proportionality constant. (For a discussion of this in the context of chemical reactions, see ref. 19.)

\[ \frac{dX_3}{dt} = k_3X_1X_2 \]

\( X_1, X_2, \) and \( X_3 \) represent the concentrations or amounts of two elements and a complex, which is formed by their association; \( k_3 \) is a proportionality factor that is constant when all else remains unchanged. This type of description is the well-established law of mass action in chemical and biochemical kinetics and, as is well known, it can be applied to populations of organisms in ecological systems.

In many cases, the mechanism may involve several elements and several elementary associations (and/or dissociations). The rate of production of an output can then be represented by a polynomial function composed of input, output, and intermediate variables:

\[ \frac{dX_m}{dt} = \left[ P_m(X_1, X_2, \ldots, X_i; X_i, \ldots, X_k; X_i, \ldots, X_n) \right] \]

[1]

or, in certain cases (for a discussion in the context of biochemical kinetics, see ref. 19), by a rational function composed only of inputs and output variables

\[ \frac{dX_m}{dt} = \left[ R_m(X_1, X_2, \ldots, X_i; X_i, \ldots, X_n) \right] \]

[2]

Eqs. 1 and 2 are general and can accurately represent the phenomena of interest, but they are mathematically much too complicated to be of use in real systems that are composed of many such mechanisms. Some form of simplifying approximation is required. A linear approximation is generally not sufficient for synergistic systems; for these systems the approximation must be nonlinear and yet simple enough to treat mathematically.

Polynomial and rational functions can be approximated over a wide range of values for the variables by a linear relationship in a space with logarithmic coordinates (19, 20); this approximation corresponds to a product of power-law functions in the conventional space with cartesian coordinates (19). Therefore,

\[ \frac{dX_m}{dt} = \alpha_m \prod_{p=1}^{n} X_p g_{mp} \]

in which \( \alpha_m \) and \( g_{mp} \) are parameters whose values depend upon the range of values being approximated. The validity of this approximation rests on theoretical grounds and is guaranteed, at least over a certain range of values for the variables. However, indications of its validity also can be obtained by directly observing the component mechanisms of complex systems. I shall consider just three examples from widely different types of systems.

Koren et al. (21) have examined the kinetics of individual biochemical reactions within a living cell and found that these are best described by power-law functions of the reactant concentrations. In this context, the parameter \( \alpha_m \) can be associated with the apparent rate constant of the reaction in question, and the parameters \( g_{mp} \) can be identified as the apparent kinetic orders of the reaction with respect to the various reactants (19).

For sensory neurons there is generally a nonlinear relationship between the rate or frequency of discharge and the magnitude of sensory input or stimulus. Rosner and Goff (22) and Stevens (23) reviewed data from a large number of studies dealing with many different sensory modalities and showed that the nonlinear relationship is a power law in nearly every case.

Cobb and Douglas (17) proposed that the rate of production for an industry or a sector of the economy could be related to the various factors of production by power-law functions. Subsequent studies have shown the widespread applicability of this form of production function (e.g., see refs. 18, 24, and 25), although other production functions have been proposed. The parameters \( g_{mp} \) in the Cobb–Douglas production function are called elasticities with respect to the various factors of production. Power laws also are often used for consumption or demand functions because they fit historical data rather well (30). The studies of demand for various English and American commodities by Stone (31) provide numerous examples.

Thus, the product of power laws provides an appropriate mathematical description for elementary synergistic mechanisms. It is a relatively simple function and yet one that retains the ability to represent the essential nonlinear features of such mechanisms.

Description of integrated systems

Theoretically, the behavior of a complex synergistic system can be related to the nature of its component mechanisms. The prerequisite for obtaining this relation is a knowledge of the

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\( ^d \) In principle, these parameters of the power law can always be related to the parameters of the corresponding polynomial or rational function.
elementary descriptions for each of the component mechanisms. The mathematical form for these descriptions was presented in the previous section for a broad class of mechanisms called synergistic. These component descriptions will now be combined to yield a set of differential equations capable of representing the intact system.

For purposes of analysis, spatially distributed systems can be conceptually subdivided into compartments sufficiently small that within them the system may be considered spatially homogeneous. The concentration or amount of an element within such a compartment will be represented by the symbol $X_i$, where the subscript $i$ signifies both the name (type) and the location (compartment) of the element. For example, the symbols $X_p, X_u, X_r$ might represent amounts of a single type of element at three different locations or they might represent amounts of three different types of elements at a single location.

The functional equations describing a general system of $n$ elements different in kind or location or both are the following:

$$
\dot{X}_i = \sum_{j=1}^{s} P_{ij}^+ - \sum_{j=1}^{s} P_{ij}^- i = 1, 2, \ldots, n, \tag{3}
$$

in which $\dot{X}_i$ represents the time derivative of $X_i$. The polynomial (or rational) functions $P_{ij}^+$ represent the rates of elementary mechanisms that increase $X_i$; similarly, the $P_{ij}^-$ represent the rates of elementary mechanisms that decrease $X_i$. In general, these polynomial (or rational) functions are dependent upon all the variables of the system—i.e.,

$$
P_{ij}^+ = P_{ij}^+(X_1, X_2, \ldots, X_n),
$$

$$
P_{ij}^- = P_{ij}^-(X_1, X_2, \ldots, X_n).
$$

Additional variables $X_{n+i}$ will be defined as aggregate measures of the entire system and of particular subsystems within the system. For example, these could be the total weight of an organism or a particular organ of the organism; the total population of a society or a particular group within the society; the capital accumulation of an economy or of a particular sector of that economy, etc. Each $X_{n+i}$ is the sum of all the relevant elements of the system or subsystem—i.e.,

$$
X_{n+i} = \sum_{j=1}^{s} \text{Kernel}_{ij} X_i i = 1, 2, \ldots, s - n. \tag{4}
$$

The corresponding rates of growth are

$$
\dot{X}_{n+i} = \sum_{j=1}^{s} \text{Kernel}_{ij} \dot{X}_i i = 1, 2, \ldots, s - n,
$$

and by using the relations in Eq. 3, these can be written

$$
\dot{X}_{n+i} = \sum_{j=1}^{s} \text{Kernel}_{ij} \sum_{k=1}^{s} P_{jk}^+ - \sum_{j=1}^{s} \text{Kernel}_{ij} \sum_{k=1}^{s} P_{jk}^- i = 1, 2, \ldots, s - n. \tag{5}
$$

Because the sum of two polynomial (or rational) functions is also a polynomial (or rational) function, Eqs. 3 and 5 can be rewritten in the form

$$
\dot{X}_i = V_i(X_1, X_2, \ldots, X_n) - V_i^-(X_1, X_2, \ldots, X_n) \tag{6}
$$

in which $V_i$ is a polynomial (or rational) function representing the composite rate of increase in $X_i$ and, similarly, $V_i^-$ is a polynomial (or rational) function representing the composite rate of decrease in $X_i$. Because Eqs. 4 and 6 are composed of polynomial (or rational) functions with properties identical to those of Eqs. 1 and 2, they also can be approximated by a product of power laws. Thus,

$$
\dot{X}_i = \alpha_i \prod_{j=1}^{s} X_j^{\beta_{ij}} - \beta_i \prod_{j=1}^{s} X_j^{\beta_{ij}} i = 1, 2, \ldots, s \tag{7}
$$

and

$$
\dot{X}_i = \gamma_i \prod_{j=1}^{s} X_j^{\beta_{ij}} i = n + 1, n + 2, \ldots, s. \tag{8}
$$

Here we are using products of power laws to represent both the synergistic components, which give rise to the nonlinearity of the system (Eq. 6), and the aggregate measures, which are simple linear sums (Eq. 4). The latter may seem unusual because we are accustomed to approximating product nonlinearities by linear sums. In fact, the appropriately chosen product of power laws is a surprisingly good approximation to a linear sum over a considerable range in the values of the variables. (For a specific example, see ref. 19.)

The behavior of complex synergistic systems is determined by the nature of the component mechanisms and their abundant interrelationships, as expressed mathematically in Eqs. 7 and 8. The problem of analyzing such systems can be viewed as a process of extracting the information latent in these descriptive equations.

A basic growth equation

By starting with Eqs. 7 and 8 and a single reasonable postulate, one can derive a basic growth equation that has as special cases all the well-established growth equations. The postulate is the following: Most changes among the component parts of a system occur much "faster" than the rate of growth for the system as a whole. Mathematically, this implies that a small number ($k$) of equations in [7], representing the "slowest" phenomena, determine the temporal response of the entire system. We call these "temporally dominant"; all other equations, representing the faster phenomena, can be assumed to have reached a quasi-steady state with time derivatives equal to zero. This postulate has a long history and its justification and utility are well known. (For further discussion see refs. 19 and 32.)

By suitably renumbering the variables in Eqs. 7 and 8, one can have the $k$ temporally dominant equations first, then the quasi-steady-state equations, and the aggregate equation of the entire system last. The time derivatives of the $n - k$ variables that are in quasi-steady state can be set to zero

$$
0 = \alpha_i \prod_{j=1}^{s} X_j^{\beta_{ij}} - \beta_i \prod_{j=1}^{s} X_j^{\beta_{ij}} i = k + 1, k + 2, \ldots, n. \tag{9}
$$

From these nonlinear algebraic equations one can solve for the last $n - k$ variables of the system in terms of the first $k$ variables (33):

$$
X_i = \delta_i \prod_{j=1}^{k} X_j^{\nu_{ij}} i = k + 1, k + 2, \ldots, n. \tag{9}
$$

<table>
<thead>
<tr>
<th>Table 1. Comparison of the basic growth equation in a single variable and several special cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>Growth law</td>
</tr>
<tr>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>Basic*</td>
</tr>
<tr>
<td>Linear</td>
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<tr>
<td>Exponential</td>
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<td>Monomolecular</td>
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<td>Logistic</td>
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<tr>
<td>Bertalanffy</td>
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</table>

* $X_i = \alpha_i X_i^{g_{11}} - \beta_i X_i^{h_{11}}$.
Table 2. Comparison of the basic growth equation in two variables and several special cases

<table>
<thead>
<tr>
<th>Growth law</th>
<th>Parameters</th>
<th>Refs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic*</td>
<td>$\alpha_1$ $g_{11}$ $g_{12}$ $\beta_1$ $h_{11}$ $h_{12}$ $\alpha_2$ $g_{21}$ $g_{22}$ $\beta_2$ $h_{21}$ $h_{22}$</td>
<td>—</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>$b$ 0 1 0 0 0 0 0 0 0 1 0 2</td>
<td>47</td>
</tr>
<tr>
<td>Power law</td>
<td>$bc$ 0 $(1-b)$ 0 0 0 0 0 0 0 1 0 2</td>
<td>34, 38, 48-50</td>
</tr>
<tr>
<td>Hyperbolic</td>
<td>$b$ 0 2 0 0 0 0 0 0 0 $a^{-1/2}$ 0 2</td>
<td>51</td>
</tr>
<tr>
<td>Weibull</td>
<td>$ck$ 1 $(c+1)$ 0 0 0 0 0 0 0 1 0 2</td>
<td>52, 53</td>
</tr>
<tr>
<td>Stochastic</td>
<td>$2a_k$ 1 $1/2$ 0 0 0 0 0 0 0 1 $1/2$</td>
<td>54, 55</td>
</tr>
<tr>
<td>Gompertz</td>
<td>1 1 1 0 0 0 0 0 0 $c$ 0 1</td>
<td>34, 38, 43, 44, 56, 57</td>
</tr>
<tr>
<td>Lotka–Volterra</td>
<td>$r$ 1 0 $k$ 1 1 $K$ 1 1 0</td>
<td>14, 15</td>
</tr>
</tbody>
</table>

* $\dot{X}_i = \alpha_1 X_i X_j^{g_{1j} h_{1j}} - \beta_1 X_i^{h_{1j}} X_j^{g_{1j}}$, $\dot{X}_2 = \alpha_2 X_i X_j^{g_{2j} h_{2j}} - \beta_2 X_i^{h_{2j}} X_j^{g_{2j}}$.

in which $\delta_i$ is a function of the original $\alpha$, $\beta$, $g$s, and $h$s, and $e_i$ is a function of the original gs and hs.

Eq. 9 can be substituted into Eqs. 7 and 8, and the resulting equations will be of the form:

$$\dot{X}_i = \alpha_i \prod_{j=1}^{k} X_j^{\delta_{ij} h_{ij}} - \beta_i \prod_{j=1}^{k} X_j^{h_{ij}}$$

$$i = 1, 2, \ldots, k; n + 1, n + 2, \ldots, s \quad [10]$$

and

$$X_i = \gamma_i \prod_{j=1}^{k} X_j^{\delta_{ij}}$$

$$i = n + 1, n + 2, \ldots, s \quad [11]$$

Although the same symbols have been used for notational simplicity, it should be noted that the $f$s, $g$s, and $h$s in Eqs. 10 and 11 are actually functions of the original $f$s, $g$s, and $h$s in Eqs. 7 and 8. Similarly, the $\alpha$s, $\beta$s, and $\gamma$s in Eqs. 10 and 11 are functions of all the original parameters in Eqs. 7 and 8.

One can invert the last equation in [11] to give $X_i$ as a product of power laws involving $X_2$, $X_3$, \ldots, $X_k$ and $X_i$. Substitution of this inverted relation into the remainder of Eqs. 10 and 11 allows one to write the following set of equations

$$\dot{X}_i = \alpha_i X_i^{g_{i1} h_{i1}} \prod_{j=2}^{k} X_j^{g_{i2} h_{i2}} - \beta_i X_i^{h_{i1}} \prod_{j=2}^{k} X_j^{h_{i2}}$$

$$i = 1, 2, \ldots, k \quad [12]$$

in which, again, the parameters only indicate the form of the equations and it must be remembered that they are actually functions of the original parameters in Eqs. 7 and 8.

Finally, the numbers of the first and the last variables (i.e., 1 and s) can be interchanged so that the aggregate measure for the entire system becomes $X_1$. Then Eq. 12 can be written

$$\dot{X}_i = \alpha_i \prod_{j=1}^{k} X_j^{\delta_{ij} h_{ij}} - \beta_i \prod_{j=1}^{k} X_j^{h_{ij}}$$

$$i = 1, 2, \ldots, k \quad [13]$$

Note that this equation has exactly the same form as the original Eq. 7, except the number of rate-determining variables is $k$ rather than $n$. This telescoping of the equations into a smaller number of equations with exactly the same form is a very important property of any formalism for the analysis of complex systems. Most formalisms do not have this property, but the synergistic formalism described in the previous section and the linear formalism do.

Discussion

Eq. 13 can be defined as a generalized growth equation in differential form for the entire system when there are $k$ temporally dominant processes. However, most systems appear to be governed by a very small number of temporally dominant equations. In fact, all the well-known growth equations are special cases of Eq. 13 when there are only one or two temporally dominant equations (e.g., see Tables 1 and 2), as I shall show elsewhere (58).

The generality of the basic growth equation [13] is further indicated by a curious relationship to probability distribution functions. Any system that grows into a stable mature form has a growth curve that is a legitimate cumulative probability distribution. Conversely, the integral function of any probability distribution exhibits properties of limited growth. The extent to which these two concepts are interchangeable remains to be explored. However, it is already clear that a number of well-known probability distributions (e.g., uniform, Gaussian, Rayleigh, Maxwell, Pearson) can be represented by the basic growth equation [13].

It should be clear from the foregoing development that this basic growth equation is not simply another empirically derived formula but is based upon the nature of the elemental mechanisms in synergistic systems. In principle, this theory allows overall properties of growing systems to be related to the parameters of the underlying processes. However, because most systems of interest are complex, it may be some time before this program is carried out and such relations are made explicit. This need not concern us, for we shall be more interested in the numerous testable predictions that follow naturally from this theory.

Statistical mechanics provides an appropriate analogy here. Within this theoretical framework one is able to derive macroscopic properties, such as the gas laws, from the kinetic behavior of the individual molecules that comprise the system. The fact that in practice one cannot accurately observe the molecules, record their velocity, and keep track of all their collisions is unimportant for the theoretical development (59).

Similarly, the primary importance of the foregoing development is that it provides a unified theoretical framework, precisely what has been lacking in previous treatments of growth, within which most of the well-known properties of growing systems can be derived. These derivations will be the subject of subsequent publications (8, 58).

I thank Robert Rosen and A. M. Kotze for helpful criticisms of the original manuscript. This work was done in 1976–1977 while I was a recipient of a fellowship from the John Simon Guggenheim Memorial Foundation. It also was supported in part by a grant from the National Science Foundation (BMS 75-01591).

5. Pareto, V. (1897) Cours d'Économie Politique, Lausanne [reprinted by Librairie Croz, Genève, Switzerland (1964)].


42. Pearl, R. (1924) Studies in Human Biology (Williams & Wilkins, Baltimore).


