On the asymptotic eigenvalue distribution of a pseudo-differential operator

(uncertainty principle/canonical transformations/packings of unit cubes)

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Abstract

A description of the number $N(K)$ of eigenvalues less than $K$ for a pseudo-differential operator with positive symbol is given in terms of the number of unit cubes canonically imbedded in the subset of phase space where the symbol is less than $CK$. This gives back in particular the order of magnitude of $N(K)$ for elliptic symbols.

This paper is devoted to a description of the growth of the number $N(K)$ of eigenvalues less than $K$ of a pseudo-differential operator with positive symbol. Very precise information on $N(K)$ has previously been obtained by various authors under more restrictive conditions, notably by Hörmander (1) for elliptic operators and by Menikoff and Sjöstrand (2) for certain classes of subelliptic operators with loss of one derivative.

We first state our main results for symbols which have already been localized in phase space. Thus, let $a(x,\xi) \in C^\infty(R^n \times R^n)$ be a positive function satisfying the inequalities

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} M^{2-|\beta|} \quad \text{for} \quad (x,\xi) \in R^n \times R^n \quad [1]$$

$$a(x,\xi) \geq CM^2 \quad \text{when} \quad \max\{|x|,|\xi|/M| \geq 1 \quad [2]$$

let $a(x,D)$ be the corresponding pseudo-differential operator, and denote by $S(a,K)$ the set

$$S(a,K) = \{(x,\xi) \in R^n \times R^n; \quad a(x,\xi) < K \}.$$

It was shown in ref. 3 that the first eigenvalue of $a(x,D)$ is bounded below (up to multiplicative constants) by the first $K$ for which $S(a,K)$ contains the image of the unit cube in phase space by a canonical transformation with suitable bounds; this then raises the question of whether $N(K)$ can be compared to the number in $S(a,K)$ of disjoint images of the unit cube by canonical imbeddings. The following theorem provides an answer to this question.

Theorem 1. There exists an algorithm, "a canonical packing", associating to each $K(C,M^* \leq K \leq M^2)$ a set $Q^0(a,K)$ of disjoint images of the unit cube by canonical transformations, which are all contained in $S(a,C,K)$ and have the following property:

- If we define

$$L(K) = \text{number of elements in } Q^0(a,K)$$

$$N(K) = \text{number of eigenvalues } <K \text{ of the quadratic form}$$

$$\text{Re}(a(x,D)u,u)$$

then

(A) $N(c_0 K) \leq C_0^2 L(K)$

(B) $N(c_1 K) \geq c_1 L(K)$.

Here $c_0,c_1,c_2,c_3$ are constants depending only on $\epsilon$ and the dimension $n$.

To state global results, consider a compact manifold $X$ equipped with a positive smooth measure $\mu$. If $a(x,\xi)$ is a positive classical second-order symbol defined on $T^*(X)$, then the quadratic form $\text{Re}(a(x,D)u,u)_{L^2(\mu)}$ is specified modulo errors $O(|u|^3)$. Thus to ensure $\text{Re}(a(x,D)u,u)$ has discrete eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ tending to infinity we assume the subelliptic estimate

$$\text{Re}(a(x,D)u,u) + C\|u\|^2 \geq c \|u\|_0^2$$

where $\|u\|_0^2$ denotes a Sobolev norm. Note that ref. 5 allows us to check whether the above estimate holds for a given positive $a(x,\xi)$ and that the order of magnitude of $\lambda_k$ is independent of the choice of $\mu$.

A slight modification of "the canonical packing" of Theorem 1 produces a family $Q(a,K)$ of pairwise disjoint canonical images of the unit cube, all contained in $(x,\xi) \in T^*(X); a(x,\xi) < K$. Set $L(K)$ equal to the number of cubes in $Q(a,K)$, and set $N(K)$ equal to the number of eigenvalues $< K$. Then

Theorem 2. Under the above hypotheses there exist constants $K_0,C,C',c'$ for which $N(cK) \leq C L(K), N(c'K) \geq c' L(K)$, whenever $K \geq K_0$.

It is not difficult to deduce Theorem 2 from Theorem 1, so we confine our further discussion to the local result.

We now describe the algorithm mentioned above and sketch a proof of Theorem 1. The arguments involved rely heavily on the $S^m_\phi$ symbolic calculus of Beals and Fefferman (3), the sharp Gårding inequality with gain of two derivatives of ref. 4, and the microlocalization procedures of ref. 5.

The construction of $Q^0(a,K)$ is by induction on the dimension $n$. If $n = 0$, $a$ is a real number, phase space consists of a single point $z_0$, and we set

$$Q^0(a,K) = \begin{cases} 
\phi & \text{if } a \leq K \\
\{z_0\} & \text{if } a > K.
\end{cases}$$

Assuming the algorithm for $Q^{n-1}(a,K)$ is known, $Q^n$ is obtained as follows.

Start with $B = \{(x,\xi) \in R^n \times R^n; |x|,|\xi|/M \leq 1\}$. Cut $B$ dyadically, retaining those $B_j$'s for which one of the following occurs:

(a) $a \geq cM_j^2$ on $B_j$^****

(b) $M_j^2 \leq cK$ but $a$ is false

(c) $a$ and $b$ are false, and

$$\max |D_x^\alpha D_\xi^\beta a(x,\xi)| \geq C \max_{|\alpha|+|\beta|=2} M_j^{\delta_j}.$$**

Here we have denoted by $\delta_j \times M_j \delta_j$ the sides of $B_j$, by $B_j^*$ the dilate of $B_j$ by a large constant, and written $M_j$ for $M_j^2$.

Let $J_0,J_1,J_2$ be the sets of indices $j$ corresponding to the different types of boxes $B_j$ above. Given a box $B$, it is convenient to introduce $S^m(B)$ as the space of all $C^m$ functions $p(x,\xi)$ satisfying

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_0(|\text{diam}_x B| + |\text{diam}_\xi B)|^{m-|\alpha|}(\text{diam}_x B)^m - |\delta|.$$**

When $B$ is the form $1 \times M$ we shall also write $S^m(M)$ for $S^m(B)$.

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For each $B_j$ we now define a collection $Q_j$ of canonical images of the unit cube as follows:

For $j \in J_a$ take $Q_j = \emptyset$.

For $j \in J_b$, take $Q_j$ to be the set of all pieces obtained by cutting $B_j$ into equal blocks of sizes $M_j^{1/2} \times M_j^{1/2}$.

For $j \in J_c$, we first find a canonical transformation $\Phi_j: B_j^{**} \rightarrow B_j^*$ with $B_j^*$ having sizes $1 \times M_j$ and a superset $\Phi_j = \xi_j + p_j(x, \xi_j)$ on $B_j^*$ ($\xi_j = (\xi_j, \ldots, \xi_n)$). Choose symbols $\theta_j(x, \xi_j) \in S^0(M_j^2)$ with $0 \leq \theta_j \leq 1$, $\theta_j = 1$ if $|x|^2 + |\xi_j|^2 / M_j \leq 100$, $\theta_j = 0$ if $|x|^2 + |\xi_j|^2 / M_j \geq 500$, and define

$p_j(x, \xi_j) = K^{1/2} \theta_j(x, \xi_j) \sum p_j(x, \xi_j) dx_1 + M_j^{1/2} |1 - \theta_j(x, \xi_j)|$

for each integer $j$ satisfying $|j| \leq 2K^{1/2}$. We now let $Q_j$ be the collection of regions of the form $(x_1, \xi_1); |x_1 - \ell K^{1/2}| < K^{1/2}/2, |\xi_1| < K^{1/2}/2 \times Q$ where $|j| \leq 2K^{1/2}$ and $Q \in Q^n = \{p_j(x, \xi_j)\}$ and set

$Q_j = \{\Phi_j(Q); \ Q \in Q_j\}$

For each $j$, $Q_j$ is then a pairwise disjoint collection of canonical images of the unit cube contained in $S(a, CK) \cap R^M$. Next let $\#(B_j) = \text{number of elements in } Q_j$, and define a finite sequence $B_{j-1}, \ldots, B_{j_r}$ by picking $B_j$ to maximize $\#(B_j)$, and $B_{j_r}$ to satisfy $B_{j_{r-1}} \cap B_j = \emptyset$ for $r \leq s$, with $\#(B_{j_s})$ as large as possible. The sequence $\{B_j\}$ eventually stops since the set of all $B_j$ is finite. The $B_j$'s are then pairwise disjoint and $\bigcup_{i=1}^n Q_i \bigcap B_j = \emptyset$; to see this associate to each $B_j$ the first $B_j$ with $B_j \cap B_j = \emptyset$. We then have $\#(B_j) \geq \#(B_j)$, while each $B_j$ is paired to a bounded number of $B_j$ since $Vol(B_j) \sim Vol(B_j)$ and $B_j \subset B^{**}$. Finally, we set

$Q^n(a, K) = \bigcup_{s=1}^n Q_{i_s}$

which is then a collection of pairwise disjoint canonical images of the unit cube all contained in $S(a, CK)$. Observe that the number of elements in $Q^n(a, K)$ is $\sim \sum \#(B_j)$. The construction is complete.

Proof of Part A of Theorem 1. To prove A, it will suffice to construct a subspace $H$ of codimension $\leq C_a(L(K))$ in $L^2(R^N)$ such that

$Re(a(x, D)u, u) \geq c_K \|u\|^2$ for all $u \in H$. [3]

Let $\tau \in S^0(M)$, $\sigma_j, \psi_j \in S^0(B_j)$ be symbols satisfying $\tau = 0$ in $B^{**}$, $\tau = 1$ outside of $B^{**}$, $1 = \tau + \sum \sigma_j$, supp $\sigma_j \subset B_j$, supp $\psi_j \subset \text{supsupp } \sigma_j$, $\psi_j \equiv 1$ on supp $\sigma_j$.

For $B_j$ satisfying $b$, denote by $(x_j, \xi_j)$ the center of $B_j$ and let $[\sigma_j^{(*)}]$ be the collection of eigenfunctions with eigenvalues $\leq AK$ of the Hermite operator

$H_j(x, \xi) = (M_{\delta_j})^2|x - x_j|^{2} + \delta_j^2|\xi - \xi_j|^2$.

Here $\delta_j$ is a large constant to be determined later.

For $B_j$ satisfying $c$ and $\delta$ integer, $|\ell| \leq 2K^{1/2}$, let $[\sigma_j]\}$ be the collection of eigenfunctions with eigenvalues $\leq C_K$ of the quadratic form $Re(p(x, D)\psi_j, \psi_j)$, and let $U_j$ be Fourier integral operators such that

$Re(q(x, D)u, u) = Re(q \psi_j(x, \xi_j)u, u) + 0(\|u\|^2)$

for $q \in K^2(B_j)$ with support included in supp $\sigma_j$. The existence of $U_j$ is guaranteed by the sharp Egorov principle, and the symbol of $U_j$ may be assumed to be supported in

$|\ell| \leq 2K^{1/2}$.

The space $H$ is then the space of all $u \in L^2(R^N)$ which satisfy the two sets of conditions:

$Re(a(x, D)u, u) \geq c_K \|u\|^2$ for $\ell \in J_b$

$Re(a(x, D)u, u) + 0(\|u\|^2)$

$\leq c_K \|u\|^2$ for $\ell \in J_c$.

Since the number of $|\sigma_j|$ in case $b$ is $\sim \#(B_j)$ and the number of $|\sigma_j|$ in case $c$ is $\leq C \#(\#(B_j) \cap CK)$, the inductive hypothesis follows that the codimension of $H$ is $\leq CL(K)$.

The proof of [3] will be easy once the following estimates have been established.

$Re(a(x, D)u, u) \geq c_K \|u\|^2 - C_{M-N} \|u\|^2$, [4]

for $\ell \in J_a \cup J_c$, $u \in H$.

$\|u\|^2 \leq \left(\sum_{j \in J_b} \|\sigma_j(x, D)u\|^2 + \|\tau(x, D)u\|^2\right)^{1/2} u \in H$. [5]

In fact, [4], [5], the ellipticity of $a(x, \xi)$ outside of $B$, and the sharp Garding inequality applied to the boxes $B_j$ of case $b$ together imply:

$Re(a(x, D)u, u) \geq c_K \left(\sum_{j \in J_b} \|\sigma_j(x, D)u\|^2 + \|\tau(x, D)u\|^2\right)^{1/2} - c \sum_{j \in J_b} \|\sigma_j(x, D)u\|^2 - C \|u\|^2 \geq c'K \|u\|^2$, $u \in H$,

which is the desired inequality.

To prove [5] observe that if $j \in J_b$ and $u \in H$ then

$\sum_{j \in J_b} Re(H_j(x, D)\sigma_j(x, D)u, \sigma_j(x, D)u) \geq AK \sum_{j \in J_b} \|\sigma_j(x, D)u\|^2$

[6]

in view of the conditions (C); since the left-hand side of [6] is $Re(\sum_{j \in J_b} H_j\sigma_j^2(x, D)u, u) + 0(\|u\|^2)$ and $(\sum_{j \in J_b} H_j\sigma_j^2(x, \xi) \leq CK$, this implies

$\sum_{j \in J_b} \|\sigma_j(x, D)u\|^2 \leq C(a, C', A)$,

which for $A$ large yields [5] as a simple consequence. As for [4], it is trivial for $j \in J_a$ since $a(x, \xi)$ is then elliptic, and we need only consider the case $j \in J_c$. Thus, let

$q_j(y, \eta) = \theta_j(y', \eta') \psi_j(y', \eta') + M_j^{1/2} \{1 - \theta_j(y', \eta')\}$

$v_j = U_jq_j(x, D)u$

$v_j(y) = K^{1/2} \sum |v_j(y')dy'|, \ |\ell| \leq 2K^{1/2}$

$v_j(y) = (y_1 + rK^{-1/2}, \ldots, \eta' \leq L, L \gg 1, \text{and } \gamma_M(\eta) \text{ be a C}_a \text{ function equal to } \gamma_M \text{ for } |\eta| \leq M_j, \text{ to } M^2 \text{ for } |\eta| \geq 2M_j \text{ with good bounds}; we then have

$Re(a(x, D)\sigma_j(x, D)u, \sigma_j(x, D)u) \geq c \gamma_M(\eta) v_j(y) v_j(y')$ + $c \gamma_M(\eta) v_j(y) v_j(y')$ - $C \|\sigma_j(x, D)u\|^2$,

where $c \gamma_M(\eta) v_j(y) v_j(y')$ - $C \|\sigma_j(x, D)u\|^2$. [7]

The spectral decomposition theorem of [5] now shows that up
to multiplicative constants the integrand for each $j, \ell$ is bounded below by

$$\inf_{w \in L^2(\mathbb{R}^{n-1})} |K\|w - \xi_{j,\ell}\|^2 + \Re\{\sum_{|\ell| \leq 2K/\epsilon} q_1(y_1 + rK^{-1/2}/L, y', D') |w, w')\}$$

which is in turn greater than

$$c \inf_{w \in L^2(\mathbb{R}^{n-1})} |K\|w - \xi_{j,\ell}\|^2 + \Re\{p_1(y', D') |w, w')\}, \quad c > 0 \quad [9]$$

since $q_1(y, \eta')$ is non-negative and $p_1(y', \eta')$ is just the average of $q$ over the interval $[y_1 - \ell, y_1 - K^{-1/2}] < K^{-1/2}/2$ for each $(y', \eta')$. However, conditions $(C^1)$ simply say that $(\xi_{j,\ell}, \xi_{j,\ell}') = 0$ and thus

$$\inf_{w \in L^2(\mathbb{R}^{n-1})} |K\|w - \xi_{j,\ell}\|^2 + \Re\{p_1(y', D') |w, w')\} \geq cK |\|\xi_{j,\ell}\|^2 \quad [10]$$

In view of $(7)$, $(8)$, $(9)$, and $(10)$, we can now write

$$\Re\{\langle a(x, D)\xi_j(x, D)u, \xi_j(x, D)u \rangle \} \geq cK \sum_{|\ell| \leq 2K/\epsilon} K^{-1/2} |\|\xi_{j,\ell}\|^2 - C |\|\xi_j(x, D)u\|^2,$$

which, together with the estimate

$$\Re\{\langle a(x, D)\xi_j(x, D)u, \xi_j(x, D)u \rangle \} \geq cK \sum_{|\ell| \leq 2K/\epsilon} \int_{[y_1 - 1, y_1 + 1]} |\xi_{j,\ell}\|^2_{L^2(\mathbb{R}^{n-1})} dy_1 - C |\|\xi_j(x, D)u\|^2,$$

and the fact that $\xi_j$ is supported in $|y_1| \leq 2$, yield

$$\Re\{\langle a(x, D)\xi_j(x, D)u, \xi_j(x, D)u \rangle \} \geq cK |\|\xi_{j,\ell}\|^2 - C |\|\xi_j(x, D)u\|^2.$$

The desired estimate follows by applying the symbolic calculus and the theorem of Ergorov. The proof of A is complete.

Proof of Part B of Theorem 1. Recall that $L(K) = \sum_{\#(B_j)}$ if $j \neq j'$. For $j \leq j'$ let $W_j$ be Fourier integral operators such that

$$\Re\{\langle q(y, D)\xi_j(x, D)W_j(x, D)\xi_j(x, D)u, \xi_j(x, D)u \rangle \} = 0(\|\xi_j\|^2),$$

for $q \in S^0(M_{j_1})$ supported in a dilate of $|y_1 + |\eta|/M_{j_1} \leq 1$. Let $\phi \in C^\infty_c(\mathbb{R})$ be a fixed function supported in $|t| \leq 1/2$ which does not vanish identically and define

$$\hat{\phi}(y) = \phi(-t + K^{1/2}y_1) \quad |t| \leq 2K^{1/2} \quad \hat{\phi}(y) \in \hat{\Theta} \quad \hat{\phi}(y) \in \Theta \quad \text{satisfies eigenvalues with eigenvalue} \quad \leq K/A \quad \text{of } H_j(x, D) \quad (j \neq j')$$

$$H_j = \Theta \text{ eigenspaces with eigenvalues} \quad \leq K/A \quad \text{of } H_j(x, D) \quad (j \neq j')$$

$$H_j = \left\{ \sum_{|\ell| \leq 2K/\epsilon} \hat{\phi}(y_1) \xi_\ell(y') \right\}$$

$$\leq AK \text{ of } \Re\{P_{j,\ell}(y', D') |\|\xi_j(x, D)u\|^2$$

The main properties of $H_j$ are the following:

$$\|\|w\|_{L^2(\mathbb{R}^n)} \geq c \#(B_j), \quad j \leq j' \cup j'$$

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \geq c\|\xi_j\|^2$$

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \leq C\|\xi_j\|^2$$

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \geq \frac{1}{2}\|\xi_j\|^2$$

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \leq C\|\xi_j\|^2$$

$$(11) \text{ is a consequence of the inductive hypothesis } [\text{Part B of Theorem 1 in } (n - 1) \text{ variables}]. \text{ To derive } (12) \text{ and } (13) \text{ we observe that if } j \leq j' \text{ and } o \in H_j \text{ then } |D^2o| \leq C\|\xi_j\|^2 \text{ and}$$

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \leq C\|\xi_j\|^2,$$

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \leq C\|\xi_j\|^2,$$

since $q_1(y, \eta') \leq CP_{j,\ell}(y, \eta')$ for $y_1$ satisfying $|y_1 - 1/2| < K^{-1/2}/2$. Choose a function $\beta(y_1)$ with $0 \leq \beta \leq 1, \beta = 0$ if $|y_1| \leq 3, \beta = 1$ if $|y_1| \geq 5$; then it is easily seen that

$$\Re\{\langle D^2 + q_j(y, D') \rangle + M_j^2 |\|\beta(y_1)\|^2 \xi_j(x, D)u \| \leq C\|\xi_j\|^2 \quad (15)$$

from which it follows that

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \leq C\|\xi_j\|^2 \quad (16)$$

where $B$ is a large constant which may be assumed to be $\leq M_{j_2}^2/K$ (we would be in case b otherwise). In fact

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \leq C\|\xi_j\|^2$$

the sharp Egorov principle yields [13]. Finally, if $j \leq j'$ and $o \in H_j$ then $K(1 - \phi_j) \leq CH_j(x, \xi)$ and

$$\Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} \leq C\|\xi_j\|^2$$

and arguing as before shows that

$$K|\|\xi_j\|^2 \leq \Re\{\hat{\phi}_j(x, D)W_j(x, D)\xi_j(x, D)u \} + 0(\|\xi_j\|^2) \leq K|\|\xi_j\|^2$$

which implies [14].

Now, define an operator $L: H_j \rightarrow L^2(\mathbb{R}^n)$ by letting

$$Lw = \sum_{j \neq j'} \hat{\phi}_j(x, D)w_j + \sum_{j \neq j'} \hat{\phi}_j(x, D)W_j(x, D)w_j$$

for $w = \sum_{j \neq j'} (\xi_j, v_j)$. We shall show that

$$Lw = \sum_{j \neq j'} \phi_j(x, D)w_j + \sum_{j \neq j'} \phi_j(x, D)W_j(x, D)w_j$$

and $Lw$ is analytic in the open ball $|\|w\|_{L^2(\mathbb{R}^n)} < 1$.

This will complete the proof of B since [17] shows that Image $(L)$ has dimension $\sum_{j \neq j'} \dim H_j \geq cL(K)$, while [17] and [18] together yield

$$\Re\{\langle a(x, D)u, u \rangle \} \leq C|\|u\|^2$$

for $u \in \text{Image } (L)$.

Now symbolic calculus and the fact that the $\phi_j(x, \xi)$ have pairwise disjoint supports imply

$$\|\|w\|_{L^2(\mathbb{R}^n)} = \sum_{j \neq j'} \phi_j(x, D)w_j + \sum_{j \neq j'} \phi_j(x, D)W_j(x, D)w_j$$

which implies [14].
in view of [12] and [14]. Similarly

\[ \text{Re}(a(x,D)Lw,Lw) \leq \sum_{j \in J_b} \text{Re}((\phi_j^a(x,D)v_j,v_j) + \sum_{j \in J_c} \text{Re}((\phi_j^a(x,D)W_j^1, W_j^2) + O(M^{-N}\|w\|^2) \]

\[ \leq CK \sum_{j \in J_b} \|v_j\|^2 + CK \sum_{j \in J_c} \|v_j\|^2 + O(M^{-N}\|w\|^2). \]

Here we have used [13] and the fact that \( \phi_j^a \leq CK \) when \( j \in J_b \). [18] follows, q.e.d.

When \( a(x, \xi) \) is an elliptic symbol of second order on a compact manifold \( X \) of dimension \( n \), it follows easily from Theorem 2 that \( N(K) \sim \text{Vol}(X)^{n/2} \), this of course is the order of magnitude given by the more precise formula of ref. 1, where sharp estimates for the error terms are also derived. It would also be interesting to relate our results to those of ref. 2 for subelliptic operators with loss of one derivative; this may require more careful considerations from symplectic geometry.

Finally, we observe that \( N(K) \) is here evaluated in terms of the number of unit cubes in a specific canonical packing of \( S(a,K) = \{ (x, \xi) : a(x, \xi) < K \} \). We may ask whether other canonical packings would lead to the same result and, in particular, whether the given packing contains essentially the maximum number of cubes that can be disjointly imbedded in \( S(a,K) \) by canonical transformations. Closely related to this question is the one of bounds for the \( k \)th eigenvalue \( \lambda_k \); it would be of interest to determine when the upper and lower bounds for \( \lambda_k \) given by Theorem 2 are comparable. We expect this usually to be the case.

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