Uniqueness of solutions of semilinear Poisson equations

(phenomenon/transition from population genetics to nucleon core)

KEVIN MCLEOD AND JAMES SERRIN

Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Contributed by James B. Serrin, July 6, 1981

ABSTRACT Let \( R^n \) denote \( n \)-dimensional Euclidean space, with \( n > 1 \). We study the uniqueness of positive solutions \( u(x), x \in R^n \), of the semilinear Poisson equation \( \Delta u + f(u) = 0 \) under the assumption that \( u(x) \to 0 \) as \( |x| \to \infty \). This type of problem arises in phase transition theory, in population genetics, and in the theory of nucleon cores, with various different forms of the driving term \( f(u) \). For the important model case \( f(u) = -u + u^p \), where \( p \) is a constant greater than 1, our results show (i) that when the dimension \( n \) of the underlying space is 2, there is at most one solution (up to translation) for any given \( p \) and \( u_0 \) and (ii) that when the dimension \( n \) is 3, there is at most one solution when \( 1 < p \leq 3 \). In both cases, the solution is radially symmetric and monotonically decreasing as one moves outward from the center. For dimensions other than 2 or 3, and indeed for the analogous cases of a real dimensional parameter \( n > 1 \), we obtain corresponding results. We note finally, again for the model case, that existence holds for \( 1 < p < (n+2)/(n-2) \), thus, there remains an interesting difference between the parameter ranges for which existence and uniqueness are established.

In this paper, we discuss the uniqueness of positive smooth solutions \( u(r) \) of the boundary value problem

\[
\begin{align*}
  u'' + \frac{n-1}{r} u' + f(u) = 0, & \quad r > 0 \\
  u'(0) = 0, & \quad u(r) \to 0 \quad \text{as} \quad r \to \infty.
\end{align*}
\]

[BP]

Here \( n > 1 \) is a constant and \( f(u) \) is an assigned function determining the particular form of the problem; by a smooth solution we mean a function of class \( C^1([0, \infty) \cap C^2(0, \infty) \). We make the following standing assumptions on the function \( f(u) \):

(i) \( f \in C^1([0, \infty); f(0) = 0 \), \( f'(0) < 0 \);

(ii) there exists an \( \alpha > 0 \) such that

\[ f(u) < 0 \quad \text{for} \quad u \in (0, \alpha), \quad f(u) \to 0 \quad \text{for} \quad u \to \alpha, \infty. \]

BP arises naturally in the study of positive classical solutions of the problem

\[
\begin{align*}
  \Delta u + f(u) = 0 & \quad \text{in} \quad R^n \ \\
  u(\cdot) \to 0 & \quad \text{as} \quad |\cdot| \to \infty.
\end{align*}
\]

[I]

Indeed, it was proved in an important paper (1) that any positive solution of problem I must be radial with respect to some origin of coordinates \( x_0 \) (and monotonically decreasing as one moves outward from \( x_0 \), at least if \( f \) is assumed to be of class \( C^{1+\alpha} \) on some interval \( [0, \gamma] \), \( \gamma > 0 \). Thus, solutions of I can be treated as solutions of BP, with \( r \) denoting the radial variable and \( n \) the dimension. The physically important cases are, of course, \( n = 1, 2, 3 \); when \( n = 1 \), both BP and I are easily solved by quadrature, and hence for simplicity we have omitted discussion of this case.

The fact that solutions of I must be radial does not itself guarantee uniqueness, for, apart from translations of solutions, the question of whether or not more than one radial function can satisfy BP is clearly unanswered by the work of Nirenberg and colleagues (1).

In practice, the equation \( \Delta u + f(u) = 0 \) arises in the study of phase transitions of van der Waals fluids (2–4), in population genetics (5, 6), and in nuclear physics. In the latter case, Takahashi (7) obtained a pair of coupled partial differential equations that Syngue (8) later reduced to I with \( f = -u + u^3 \). Similarly, Finkelstein et al. (9) obtained I with \( f = -u + u^3 \). In sections 2 and 4, we consider the case \( f = -u + u^3 \) in detail as an illustration of our conclusions.

Quite general conditions on \( f \) that ensure the existence of positive solutions of I have been given in an important series of recent papers (10–14). In particular, it is necessary that

\[
\begin{align*}
  (a) \quad & \lim_{u \to -\infty} u^{-1} f(u) = 0 \quad \text{for} \quad \ell = \frac{n+2}{n-2}, \quad \text{if} \quad 1 < n \leq 2, \\
  (b) \quad & \int_0^\beta f(\theta) d\theta = 0 \quad \text{for some} \quad \beta \in (\alpha, \infty).
\end{align*}
\]

As a special case, existence holds when \( f = -u + au^a + u^p \) and \( a \to 0, 1 < p < (n+2)/(n-2) \); this result was obtained first by Strauss (14) and (when \( a = 0, n = 3 \)) by Sansone (15) after earlier work of Nehari (16).

Uniqueness has been studied by Coffman (17) for the case \( n = 3, f = -u + u^3 \), and by Peletier and Serrin (18) when the graph of \( f \) satisfies a starlikeness condition. Here we obtain a companion theorem to the latter by replacing starlikeness by a convexity condition, which enables us to recover and generalize Coffman’s result. [Finally, we note that the uniqueness theorem given by Sansone (ref. 15, section 7.10) for the case \( n = 3, f = -u + u^3, 1 < p < 5 \), unfortunately contains an error on page 111.]

1. Main results

In this section we state our main conclusions; they will be used to discuss several important examples in sections 2 and 4. Because the proofs are rather technical, we postpone them to a later publication, giving only a brief outline of the method in section 3 (the function \( I(u; \alpha, c) \) appearing in the statement of the theorem below can be seen in one term of the principal differential identity of section 3). Our first result is the following 1-theorem.

1-Theorem. Let \( n > 1 \), and put

\[ I(u; \alpha, c) = (u - c)f'(u) - \sigma f(u). \]

Suppose that for each \( U > \alpha \) we can find constants

\[
\begin{align*}
  a, b & \in [n-2, \infty) \cap (0, \infty) \\
  b, c & \in R, \quad c, \in [0, U],
\end{align*}
\]

6592
Mathematics: McLeod and Serrin

(continuously depending on the choice of $U$) such that
\[(b - 1)(n - 3 + b) \leq 0, \quad (b - 1)(n - 3 + \tilde{b}) \leq 0, \]
\[(b - 1)(n - 2 - a) \leq 0, \quad (b - 1)(n - 2 - \tilde{a}) \geq 0, \]
\[a \left(1 - \frac{c}{U}\right) \leq a \left(1 - \frac{\tilde{c}}{U}\right)\]
and
\[l(u; \sigma, c) > 0 \quad \text{when } 0 < u < U, \quad u \neq \alpha, \]
\[l(u; \tilde{\sigma}, \tilde{c}) < 0 \quad \text{when } u > U, \]
where $\sigma = 1 + 2b/a$, $\tilde{\sigma} = 1 + 2\tilde{b}/\tilde{a}$. Then BVP has at most one solution.

The I-theorem itself is hard to apply directly, so we give now three corollary theorems, obtained from the I-theorem by appropriate choices of the parameters. The conditions of these corollaries can be easily checked for specific functions $f$ and suitable for our various applications.

In the first corollary (valid only for $n \geq 3$), we impose conditions on $f$ that allow us to choose $a = \hat{a} = n - 2$, $b = 0$, $\tilde{b} = 1$, $c = \tilde{c} = \alpha$, all independent of $U$. Specifically, we have the following result.

**Theorem 1.** Suppose $n \geq 3$, and that $f$ satisfies the conditions
\[\left(\frac{1}{u}ight)' > 0 \quad (u > 0, \quad u \neq \alpha), \]
\[\left(\frac{1}{u}ight)', \quad \frac{n}{n - 2} f(u) < 0 \quad (u > \alpha). \]

Then BVP has at most one solution.

The choice of parameters in the remaining corollaries is more involved but essentially is determined by insisting that $a = \hat{a}$, $b = \hat{b} = 1$, $c = \tilde{c}$.

When $c = \tilde{c} = 0$ we get Theorem 2.

**Theorem 2.** Suppose $1 < n \leq 2$, and that there is a $\tau > 1$ such that
\[\left(\frac{1}{u}\right)' > 0 \quad (u > 0, \quad u \neq \alpha) \]
\[\left[u \left(\frac{1}{u}\right)'ight]' < 0 \quad (u > \alpha) \]
Then BVP has at most one solution.

When $n > 2$, it is not always possible to choose $c = \tilde{c} = 0$, which leads to the extra conditions in the next theorem.

**Theorem 3.** Suppose $n > 2$, and that inequalities 3 and 4 hold for some $\tau \in (1, n/(n - 2))$. If, furthermore, the right hand side is a quadratic form in the variable $u^n$, with discriminant $n(p - 1)^2(np - 8)$. Hence, $2\tilde{f}^2 - \tilde{f}'' > 0$ if $p < 8/n$. If $n \leq 4$ and $8/n < p \leq n/(n - 2)$ then $np - (n + 4) \geq n - 4 \geq 0$, so again $2\tilde{f}^2 - \tilde{f}'' > 0$. Thus, Theorem 3 gives uniqueness when $p = n/(n - 2)$, $2 < n \leq 4$ $p < 8/n$, $4 < n < 8$.
\[f(u) = \sum_{k=1}^{n} a_k u^{p_k}, \quad (1 = p_1 < p_2 < \ldots < p_n).\]

Inequalities 3 and 4 become, respectively,
\[\sum_{k=1}^{n} a_k (p_k - \tau) u^{n-r-1} > 0 \quad (u > 0)\]
and
\[\sum_{k=1}^{n} a_k (p_k - \tau) u^{n-r-1} < 0 \quad (u > \alpha).\]

In particular, the above conditions will be satisfied (with \(\tau = p_n\)) if \(a_1, a_2, \ldots, a_{n-1}\) are all negative and \(a_n > 0\). Theorem 2 then gives uniqueness for \(n = 2\). In the case \(n > 2\), inequality 5 imposes additional relations on the various \(a_k\) and \(p_k\), but we have not been able to find simple conditions in this case.

3. Outline of the proof

In proving the results of section 1, it is convenient to introduce the initial value problem
\[\begin{align*}
v^\prime &+ n-1 \cdot r v^\prime + f(v) = 0, \quad r > 0 \\
v(0) &= v_0, \quad v^\prime(0) = 0
\end{align*}\]

We define
\[S^+ = \{v_0 \in (0, \infty) \mid \text{the solution } v(r) \text{ of IVP remains bounded away from } 0} \]
\[S^- = \{v_0 \in (0, \infty) \mid \text{the solution } v(r) \text{ of IVP satisfies } x(r_0) = 0 \text{ for some } r_0 > 0} \]
\[S^0 = \{v_0 \in (0, \infty) \mid \text{the solution } v(r) \text{ of IVP solves BVP}\}.\]

In refs. 12 and 18 it is shown that the sets \(S^+, S^-, S^0\) cover the entire interval \((0, \infty)\). Using this fact, we can prove the following result.

**Principal Lemma.** Suppose that to every solution \(u\) of BVP there corresponds an \(e > 0\) with the property that if \(v = v(r)\) is a solution of IVP with \(|v_0 - u(0)| < e\), \(v_0 \neq u(0)\), then \(v\) intersects \(u\) exactly once and, moreover, is not a solution of BVP.

Then BVP has a unique solution.

**Proof.** Suppose there were more than one solution of BVP. Denote by \(u_1\) and \(u_2\) two "nearest neighbor" solutions of BVP—i.e., for any \(v_0 \in (u_{10}, u_{20})\), the corresponding \(v(r)\) does not solve BVP. (The existence of "nearest neighbor" solutions follows from the second part of the hypothesis.) Now consider the situation shown in Fig. 1, where \(v_{01}\) is near to and below \(u_{20}\), and \(v_{02}\) is near to and above \(u_{10}\). By hypothesis, the solution \(v^\prime(r)\), using the obvious terminology, intersects \(u_2(r)\) exactly once. It is obvious then that \(v_{01} \in S^0 \cup S^-\), so by the remark before the Principal lemma we have \(v_{01} \in S^+\). Similarly \(v_{02} \in S^0 \cup S^+\), whence \(v_{01} \in S^-\). By the continuity of solutions of IVP with respect to the initial conditions, both \(S^-\) and \(S^+\) are open.

It follows now from the elementary topology of the real line that the interval \((u_{10}, u_{20})\) is not covered by \(S^- \cup S^+\). Hence, there must be at least one point of \(S_0\) in \((u_{10}, u_{20})\)—i.e., there must exist a solution \(u\) of BVP with \(u(0) \in (u_{10}, u_{20})\). But this contradicts the fact that \(u_1\) and \(u_2\) are "nearest neighbors", and the Principal lemma is proved.

We remark finally that the conditions of the Principal lemma are verified by application (not included here) of the following Principal identity.

**Principal Identity.** Let \(u\) and \(v\) be solution of IVP. Put \(Y = r^{n-1}\cdot v - u\) and \(Z = w^\prime\), where \(w = r^n(u - c)\) and \(a, b, c\) are constants. Let
\[W = \begin{bmatrix} Y & Z \\ Y' & Z' \end{bmatrix},\]
the Wronskian of \(Y\) and \(Z\), and put
\[D = \{r^{n-1} - 2a - 2bW\}'.\]

Then
\[r^{3-n + a + b}D = (b - 1)(n + 3 + b)(v - u) + r^2(f(v) - f(u)) + (v - u)f'(u)]Z + \{a(r + b(u - c)) f(u) - (1 + \frac{2b}{a}) f(u)\] + \[2r^2(b - 1)(2 - n + a)W\}

4. Appendix

In section 2 we considered the example \(f = -u + u^p\) (\(p > 1\)) and showed there that BVP has a unique solution when
This leaves open the situation for \( n \geq 8 \). Some further information can be obtained, however, from application of Theorem 1.

Condition 1 is clear from the convexity of \( f \) (recall that \( \alpha = 1 \) in the example), but condition 2 is not automatic; indeed, it can easily be seen to fail when \( n > 4 \) and \( p = n/(n - 2) \).

We put \( \sigma = n/(n - 2) \), so that 2 becomes

\[
I = (u - 1)f'(u) - \sigma f(u) = 1 + (\sigma - 1)u - pu^{\sigma - 1} + (p - \sigma)u^p < 0 \quad (u > 1).
\]

If \( p = 1 \) then \( I = 0 \). Hence condition 2 will be satisfied, at least for \( p \) close to 1, provided

\[
\frac{\partial I}{\partial p} \bigg|_{p=1} < 0 \quad (u > 1),
\]

which leads to the condition

\[
\frac{n}{n - 2} = \sigma > \sup_{u > 1} \left\{ \frac{u \log u + u - \log u - 1}{u \log u} \right\}.
\]

The quantity on the right can be calculated numerically, being approximately 1.298; putting \( \sigma = 1.298 \) gives \( n = 8.71 \). Thus, our method will give uniqueness for at least some \( p > 1 \) provided \( n < 8.71 \). For \( n > 8.71 \), the method breaks down.

In reference to Fig. 2, existence is known for all \( p < (n + 2)/(n - 2) \). On the other hand for \( 1 < n < 8.71 \), we have proved uniqueness within the shaded area, where the function \( \bar{p}(n) \) satisfies \( 8/n < \bar{p}(n) < n/(n - 2) \), \( 4 < n < 8.71 \). The point (3,3) in this diagram is the case obtained by Coffman (17). The question of uniqueness for points \((p,n)\) outside the shaded area but below the curve \( p = (n + 2)/(n - 2) \) is open. We are initiating numerical work to resolve this question.

This research was partially supported by a grant from the National Science Foundation.