A new family of algebras underlying the Rogers–Ramanujan identities and generalizations

(Euclidean Kac–Moody Lie algebras/standard modules/principal Heisenberg subalgebras/vacuum spaces)

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ABSTRACT The classical Rogers–Ramanujan identities have been interpreted by Lepowsky–Milne and the present authors in terms of the representation theory of the Euclidean Kac–Moody Lie algebra \( A_1^{(1)} \). Also, the present authors have introduced certain "vertex" differential operators providing a construction of \( A_1^{(1)} \) on its basic module, and Kac, Kazhdan, and we have generalized this construction to a general class of Euclidean Lie algebras. Starting from this viewpoint, we now introduce certain new algebras \( \mathcal{Z}_V \) which centralize the action of the principal Heisenberg subalgebra of an arbitrary Euclidean Lie algebra \( g \) on a highest weight \( g \)-module \( V \). We state a general (tautological) Rogers–Ramanujan-type identity, which by our earlier theorem includes the classical identities, and we show that \( \mathcal{Z}_V \) can be used to reformulate the general identity. For \( g = A_1^{(1)} \), we develop the representation theory of \( \mathcal{Z}_V \) in considerable detail, allowing us to prove our earlier conjecture that our general Rogers–Ramanujan-type identity includes certain identities of Gordon, Andrews, and Bressoud. In the process, we construct explicit bases of all of the standard and Verma modules of nonzero level for \( A_1^{(1)} \), with an explicit realization of \( A_1^{(1)} \) as operators in each case. The differential operator constructions mentioned above correspond to the trivial case \( \mathcal{Z}_V = (1) \) of the present theory.

In this paper, we launch a program to give explicit constructions of general standard modules of general Euclidean Lie algebras and, hence, to produce a wide variety of new realizations of these Lie algebras as algebras of operators. The first construction (1) of a Euclidean Lie algebra, namely \( A_1^{(1)} \), by differential operators on a "Fock space" and its sequel (2) for a general class of Euclidean Lie algebras turn out to be the "trivial" cases of the present theory, in a sense to be made precise below.

The discovery of the ideas presented here was motivated by a desire to understand more deeply the Lie theoretic significance of the Rogers–Ramanujan identities, continuing a program begun in refs. 3–7. In this paper, by introducing new algebras \( \mathcal{Z}_V \) associated with arbitrary Euclidean Lie algebras \( g \) and certain \( g \)-modules \( V \), we redress the problems of interpreting Rogers–Ramanujan-type identities and of explicitly constructing the modules \( V \) to the representation theory of \( \mathcal{Z}_V \). For \( g = A_1^{(1)} \), we develop the representation theory of \( \mathcal{Z}_V \) deeply enough to prove the Conjecture in ref. 6 relating the \( \delta \)-filtrations of the standard \( A_1^{(1)} \)-modules to the generalized Rogers–Ramanujan identities of Gordon, Andrews, and Bressoud. In the process, we obtain explicit bases of all of the standard and Verma modules of nonzero level and an explicit recursive description of the action of \( \mathcal{Z}_V \) and, hence, of \( A_1^{(1)} \), in each case.

The modules \( V \) under consideration may each be viewed as the tensor product of a "Fock space" with a "vacuum space" \( \Omega \), for the principal Heisenberg subalgebra \( \delta \) of \( g \). Our algebra \( \mathcal{Z}_V \), commutes with the action of \( \delta \) and, hence, preserves \( \Omega \) and acts richly enough on it essentially to "untwist" the action of \( g \) to the tensor product of two commuting actions.

A striking feature of \( \mathcal{Z}_V \) is that, in its action on \( V \), it satisfies identities that are themselves the "generating functions" of infinite systems of identities. In this paper, these identities are presented for \( g = A_1^{(1)} \).

We obtain a new proof of the main result (Theorem 1) of ref. 6, interpreting the classical Rogers–Ramanujan identities as the formula

\[
\chi(\Omega) = \sum_{n=0}^\infty \chi(\Omega_n) / \Omega(n-1)
\]

for the level 3 standard \( A_1^{(1)} \) modules, where \( \Omega_n \) designates the \( \delta \)-filtration of \( \Omega \) (6). The content of the Conjecture of ref. 6, whose proof is presented here, is that this same formula, for general standard \( A_1^{(1)} \)-modules, coincides with the generalized Rogers–Ramanujan identities of Gordon, Andrews, and Bressoud. However, we do not have an independent proof of these identities. The first section of this study is devoted to the definition and general properties of the algebras \( \mathcal{Z}_V \), in the setting of Euclidean Lie algebras as discussed in ref. 2. We also note a slight simplification of the proof of the main result of ref. 2, from the present viewpoint.

The second section contains the deeper analysis of the case \( g = A_1^{(1)} \). The details will appear elsewhere.

THE ALGEBRAS \( \mathcal{Z}_V \)

We shall introduce algebras \( \mathcal{Z}_V \) in the generality of Euclidean Lie algebras. We shall generally take the notation of ref. 2, whose results will be summarized below.

Let \( A = \{a_{ij}\}_{i,j=0}^N \) be a Euclidean generalized Cartan matrix (8, 9)—i.e., one listed in tables 1, 2, or 3 of ref. 2. Let the corresponding Euclidean Kac–Moody Lie algebra \( g = g(A) \) over \( \mathbb{C} \) (which we say is of type 1, 2, or 3, respectively) have canonical generators \( e_i, f_i, h_i, i = 0, \ldots, N \).

Define a \( Z \)-gradation of \( g \), called the principal gradation, by the conditions

\[
\deg e_i = -\deg f_i = 1, \quad \deg h_i = 0, \quad i = 0, \ldots, N.
\]

Let \( z \) (normalized as in ref. 2) span the center of \( g \), and let \( \pi: g \to g/\mathbb{C}z \) denote the canonical map. Set

\[
e = \sum_{i=0}^N e_i, \quad \delta = \pi^{-1}(\pi(g)^{\leq \delta}),
\]

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where the superscript denotes centralizer. As in ref. 2, denote by \( b_1, b_2, \ldots \) the sequence
\[
m^{(r)}_i + d^{(r)}, j = 1, \ldots, r, d = 0, 1, 2, \ldots
\]
arranged in nondecreasing order; here \( \ell \) is the type, and the positive integers \( h^{(r)}_i \) and \( m^{(r)}_i \) are the (generalized) Coxeter numbers and exponents listed in table Eq. of ref. 2. For \( \ell = 1 \), these are the classical Coxeter number and exponents of the underlying finite-dimensional simple Lie algebra; for \( \ell > 1 \), see ref. 10.

**Proposition 1 (1, 2).** The algebra \( \mathfrak{g} \) is a (principally) graded Heisenberg subalgebra of \( \mathfrak{g} \) with basis \( \left\{ x, p_i, q_i \mid i = 1, 2, \ldots \right\} \), where
\[
[p_i, p_j] = 0 = [q_i, q_j], \quad [p_i, q_j] = \delta_{ij} \mathbb{R}, \quad \text{for all} \quad i, j = 1, 2, \ldots,
\]
and
\[
deg x = 0, \quad \text{deg} \ p_i = b_i = -\text{deg} \ q_i, \quad \text{for all} \quad i = 1, 2, \ldots.
\]

We call \( \mathfrak{g} \) the principal Heisenberg subalgebra of \( \mathfrak{g} \).

**Proposition 2 (cf. refs. 6 and 7).** Suppose that \( k \in \mathbb{C}^* \). Then the map \( \mathfrak{g} \) is an \( \mathfrak{g} \)-module isomorphism. In particular,
\[
\chi(V) = F \cdot \chi(\Omega_0),
\]
where
\[
F = \prod_{j=1}^{\infty} (1 - q^j)^{-n}.
\]
The principally specialized characters \( \chi(V) \) of the standard modules \( V \) have known product expansions given by the "numerator formula" (4, 10, 12). Thus, the characters \( \chi(\Omega) \) for these modules have known product expansions. For the case \( \mathfrak{g} = \mathfrak{a}_1 \), see refs. 3 and 4 (cf. ref. 6, Proposition 3). For Verma modules, we easily obtain:

**Proposition 3.** If \( V \) is a Verma module (i.e., a universal highest weight module), then
\[
\chi(\Omega) = \prod_{n=0}^{\infty} \chi(\Omega_{n+1}/\Omega_n).
\]

Let \( V \) be a graded highest weight \( \mathfrak{g} \)-module with highest weight vector \( v_0 \). The \( \mathfrak{g} \)-filtration of \( V \),
\[
0 = V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V,
\]
is defined as follows (6): For all \( n \geq 0 \), \( V_n \) is the span of all the primary expressions \( x_1 \cdots x_n v_0 \geq 0 \), where each \( x_j \in \mathfrak{g} \) and at most \( n \) of the \( x_j \) lie outside \( \mathfrak{g} \). Each \( V_n \) is clearly graded. For \( n \geq 1 \), let \( \Omega_n \) be \( V_n \cap V_{n-1} \). Then each \( \Omega_n \) is graded, and
\[
0 = V_{-1} \subseteq \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega
\]
is a filtration of \( \mathfrak{g} \) such that \( \Omega = \bigcup \Omega_n \). Hence (6):

**Proposition 4.** We have
\[
\chi(\Omega) = \sum_{n=0}^{\infty} \chi(\Omega_n)/\Omega_{n+1}/\Omega_n.
\]

Eq. 1, which is a tautology as it stands, includes the classical Rogers–Ramanujan identities, as the cases of the level 3 standard \( \mathfrak{a}_1 \)-modules, by the detailed study of the sum side for these cases in refs. 6 and 7. The primary goal of the present paper is to introduce new algebras that implement the sum side of Eq. 1 in great generality.

The completion \( \mathfrak{C}(\mathfrak{g}) \) of \( \mathfrak{g} \) is defined to be the vector space \( \bigoplus_{n=0}^{\infty} \mathfrak{g}_n \). Note that \( g \) acts on its completion.

**Proposition 5 (1, 2).** The completion \( \mathfrak{C}(\mathfrak{g}) \) contains elements
\[
X^{(m)} = \sum_{j \in \mathbb{Z}} X_j^{(m)}, \quad m = 1, \ldots, N
\]
(subscript denoting homogeneous component) such that \( \mathfrak{g} \) has basis
\[
\{z, p_i, Q_i \mid i \geq 0; m = 1, \ldots, N; i \in \mathbb{Z}\}
\]
(see Proposition 1) and such that for certain scalars
\[
\mu_{m} \quad \nu_{m} (m = 1, \ldots, N; i > 0),
\]

\[
[p_i, X^{(m)}] = -\mu_{m} X^{(m)} \text{ and } [q_i, X^{(m)}] = \nu_{m} X^{(m)}
\]
in \( \mathfrak{C}(\mathfrak{g}) \).

Let \( k \in \mathbb{C} \) and \( \mathfrak{g} \in \mathfrak{k} \), and define the completion \( \mathfrak{C}(V) \) of \( V \) to be the vector space \( \bigoplus_{n=0}^{\infty} V_n \). Then \( g \) acts on the completion
of $V$, and every element of $C(g)$ may be viewed as a linear operator from $V$ to $C(V)$.

Assume that $k \in C^*$. For $m = 1, \ldots, N$, define the following linear operators (cf. ref. 6):

\[ E_{-m} = \exp \left( \sum_{i=0}^{\infty} \frac{\mu_{-mi}}{i!} k \right) \in \text{End} \, C(V) \]

\[ E_{+m} = \exp \left( \sum_{i=0}^{\infty} \frac{\nu_{mi}}{i!} k \right) \in \text{End} \, V, \]

where $exp$ denotes the formal exponential series, and

\[ Z^{(m)} = E_{-m} E^{(m)} E_{+m} \in \text{Hom}(V, C(V)). \]

For each $j \in \mathbb{Z}$, let

\[ Z_j^{(m)} \in \text{End} \, V \]

be the homogeneous component of degree $j$ (in the obvious sense) of $Z^{(m)}$, so that $Z^{(m)} = \sum_{j \in \mathbb{Z}} Z_j^{(m)}$.

**Definition:** Let $k \in C^*$, and let $V \in C_k$. Denote by $\mathcal{Z}_V$ or $\mathcal{Z}$ the subalgebra of $\text{End} \, V$ generated by

\[ \{ Z_j^{(m)} \mid j \in \mathbb{Z}, m = 1, \ldots, N \} \]

Note that the associations $V \mapsto Z_j^{(m)}, \, V \mapsto \mathcal{Z}_V$, etc., have obvious functorial properties.

**Propositions 1 and 2** and elementary properties of exponential series readily imply:

**Proposition 6.** The algebra $\mathcal{Z}$ centralizes the action of $\delta$ on $V$. In particular, $\mathcal{Z}$ preserves $\Omega$.

**Proposition 7.** For all $m = 1, \ldots, N$,

\[ Z^{(m)} = (E_{-m})^{-1} Z(E_{+m})^{-1} \in \text{Hom}(V, C(V)). \]

By looking at homogeneous components and applying Propositions 2, 6, and 7, we now readily obtain:

**Proposition 8.** The correspondences

\[ W \mapsto U(\delta) \cdot W \quad \text{and} \quad Y \mapsto Y \cap \Omega \]

define mutually inverse bijections between the set of $\mathcal{Z}$-submodules $W$ of $\Omega$ and the set of all $g$-submodules $Y$ of $V$. In particular, $V$ is $g$-irreducible if and only if $\Omega$ is $\mathcal{Z}$-irreducible.

Suppose now that $V$ is a graded highest weight module with highest weight vector $v_0$. We define the $\mathcal{Z}$-filtration of $\Omega = \Omega_V$ by

\[ \Omega^{(n)} = \Omega^{(n-1)} \oplus \Omega^{(n-1)} \oplus \cdots \oplus \Omega \]

by the condition that for all $n \geq 0$, $\Omega^{(n)}$ is the span of all the elements $z_1 \cdots z_n \cdot v_0$, $0 \leq i \leq n$, where each $z_i$ is one of the $Z_j^{(m)}$ ($j \in \mathbb{Z}, m = 1, \ldots, N$). We have:

**Proposition 9.** The $\mathcal{Z}$-filtration of $\Omega$ coincides with the $\delta$-filtration of $\Omega$, i.e., $\Omega^{(n)} = \Omega^{(n)}_{\delta}$ for all $n \geq 0$. In particular, $\Omega = \mathcal{Z} \cdot v_0$, and the identity $1$ for $V$ can be equivalently formulated using the $\mathcal{Z}$-filtration.

Now assume that $V$ is irreducible under $\delta$. This can occur only if either $g$ has symmetric Cartan matrix or $g = 0$ of type 2 or 3 and, in these cases, occurs if and only if $V$ is a basic $g$-module (2), i.e., a standard $g$-module of level 1. (The equivalence of $V$ as an $\mathcal{Z}$-module with the irreducible induced $\delta$-module $M_1$ defined above follows from the numerator formula, together with Theorems 1 and 2 of ref. 13 for $g = A_{2N}$ and from ref. 10 for $g = A_{2N}$. See ref. 2.) In this case, $\Omega$ is clearly one-dimensional, so that for each $m$ and each $j \neq 0$, $Z_j^{(m)} = 0$ on $\Omega$ and, hence, on $V$. Also, $Z_0^{(m)} = c_m$ on $\Omega$ and, hence, on $V$ for some $c_m \in \mathbb{C}$. Thus, $Z^{(m)} = c_m$ and we obtain the main result of ref. 2 by a slightly shorter argument:

**Proposition 10.** (1, 2). If $V$ is a basic $g$-module, then there are (nonzero) scalars $c_1, \ldots, c_N$ such that

\[ X^{(m)} = c_m (E_{-m})^{-1} (E_{+m})^{-1} \]

for each $m$.

As in refs. 1 and 2, we now may view $\delta$ and the $X^{(m)}$, and hence $\delta$, as differential operators.

**THE STANDARD MODULES FOR $A_1^{(1)}$**

Now we specialize to the case $g = A_1^{(1)}$. We obtain explicit bases of all the standard modules and Verma modules of nonzero level, and an explicit description of the action of $A_1^{(1)}$ in each case. As noted above, the original construction of $A_1^{(1)}$ on its basic module through differential operators (1) amounts to the trivial one-dimensional case $\dim \, \Omega = 1, \, \dim \, \mathcal{Z} = 1$, of the present theory.

Because $N = 1$, we write $Z$ for $Z^{(m)}$.

**Theorem 1.** Let $k \in C^*$ and $V \in C_k$. For an indeterminate, $z$, set

\[ Z(z) = \sum_{j \in \mathbb{Z}} z^j z, \]

a formal Laurent series in $z$ with coefficients in $\text{End} \, V$. Let $\xi_1, \xi_2$ be two commuting indeterminates. We have

\[ (1 - \xi_1 \xi_2^{-k}/(1 + \xi_1 / \xi_2) - \xi_1 \xi_2)Z(\xi_1)Z(\xi_2) = k \sum_{j \in \mathbb{Z}} j(-\xi_1 / \xi_2)^j. \]

The coefficients of $Z(\xi_1)Z(\xi_2)$ and $Z(\xi_2)Z(\xi_1)$ on the left-hand side of Eq. 2 are to be understood as formal power series in $\xi_1 / \xi_2$ and $\xi_2 / \xi_1$, respectively. The right-hand side is a formal Laurent series in $\xi_1 / \xi_2$. Eq. 2 is to be interpreted as the "generating function" for the infinite system of identities obtained by equating the coefficients of all the monomials $\xi_1^i \xi_2^j$ (i, j, $\xi_1, \xi_2 \in \mathbb{Z}$) on the two sides. Each such identity involves formal infinite sums of endomorphisms of $V$, and each such sum acts as a well-defined endomorphism of $V$ in view of the fact that the grading of $V$ is truncated from above. Theorem 1 may thus be reformulated as follows:

**Theorem 2.** Define the numbers $1 = a_0, a_1, a_2, \ldots$ by the expansion

\[ (1 - \xi_1 \xi_2^{-k}/(1 + \xi_1 / \xi_2)^{-2k}) = \sum_{j = 0}^{\infty} a_j \xi^j \]

(\$z$ an indeterminate). Then for all $\ell, \, m \in \mathbb{Z}$ with $\ell \neq 0$,

\[ \sum_{j = 0}^{\infty} a_j (Z_{m-\ell} - Z_{-m+\ell}) = 0 \]

and

\[ \sum_{j = 0}^{\infty} a_j (Z_{m-\ell} - Z_{-m+\ell}) = (-1)^{km}. \]

**Remark:** It is easy to see that for $j \geq 1$,

\[ a_j = -(4/k)_{2j} (1 + (2/k), 1 - j; 2, 2), \]

where $\phi^2$ is the hypergeometric function.

**Proposition 3** asserts that if $V$ is a Verma module, then

\[ \chi(\Omega, x) = \sum_{j \in \mathbb{Z}} p_j x^j \]

where $p$ is the classical partition function. Using this and Theorem 1 or 2, we can prove:
Suppose the and where the write Given an also for Vas the identities with highest weight of each identity of A^{(1)} of for Vas the module of level 0. For Vas the module of level 0, we have obtained a new explicit construction of A^{(1)} for each Verma module of nonzero level. Theorem 3. Let V be a Verma module of level k ∈ C* and highest weight vector v_0. Then Ω_v has basis

\[ \{ Z_0, Z_k - Z_{k-1} \cdot v_0 \}_{0 \leq i_1 \leq i_2 \leq \ldots \leq i_n < 0}, \]

with the identities 2, or equivalently 3 and 4, giving effective recursive definitions of the action of the generators of Ω_v with respect to this basis.

Remark: Combining Theorem 3 with Propositions 2 and 7, we observe that we have obtained a new explicit construction of A^{(1)} for each Verma module of nonzero level.

Remark: For V as in Theorem 3, Eq. 1 coincides with the well-known identity

\[ \prod_{j=1}^{n} (1 - q^j)^{-1} = \sum_{n \geq 0} q^n/(1 - q) \langle 1 - q^2 \rangle \langle 1 - q^3 \rangle \langle 1 - q^4 \rangle \]

of Euler.

The standard A^{(1)} modules require a much deeper analysis. In what follows, the symbols and formulas are to be interpreted as in the discussion after Theorem 1.

Let \( \xi_1, \xi_2, \ldots \) be a sequence of commuting indeterminates. For \( i > j \), let \( i \neq j \), set

\[ [ij] = (1 + \xi_j/\xi_i)(1 - \xi_j/\xi_i)^{-1}, \]

\[ C_{ij} = \frac{ij}{ij} + [ij], \]

\[ D_{ij} = \frac{ij}{ij}^{-1} + [ij]^{-1}. \]

For \( i \leq j \), define

\[ M_y = \prod_{1 \leq i < j} [\ell_i] \prod_{i < m \leq j} [im], \]

\[ M_j = \prod_{i=1}^{j} \frac{(j-i+1)}{(j-i)} M_y, \]

\[ N_j = \prod_{1 \leq \ell < m \leq j} [\ell m], \]

\[ C_{ij} = C(i,i+1)C(i+1,i+2) \cdots C(j-1,j), \]

\[ D_{ij} = D(i,i+1)D(i+1,i+2) \cdots D(j-1,j). \]

Also define

\[ C_{ij} = C(i,j), D_{ij} = D(i,j). \]

Given an expression \( f(\xi_1, \ldots, \xi_d) \) involving the first \( i \) indeterminates, write

\[ \sigma_i f = \sigma_i f(\xi_1, \ldots, \xi_d) = (1/\ell) \sum_{w} f(\xi_{w(1)}, \ldots, \xi_{w(\ell)}), \]

where the sum ranges over the symmetric group of \( \{1, \ldots, \ell\} \), and set

\[ f = f(\xi_1, \ldots, \xi_d) = f(-\xi_1, \ldots, -\xi_d). \]

Theorem 4. Let V be a standard A^{(1)}-module of level k > 0, with highest weight \( \lambda \). Let \( e = (-1)^{\lambda h_0} \) and \( t = [k/2] + 1 \).

Set \( a_0 = 1 \), and for \( j > 0 \) define

\[ a_j = a_j(\xi_1, \ldots, \xi_d) = 2s_{\lambda} J_N^{-2bk} \prod_{i=1}^{j} Z_0(\xi_i), \]

(when the noncommutative product is \( Z(\xi_i)^{-1} Z(\xi_i) \)), so that also

\[ a_j = 2s_{\lambda} J_N^{-2bk} \prod_{i=1}^{j} Z(-\xi_i). \]

Suppose that \( k \) is odd, and define the polynomial

\[ Q(v) = v^t + \sum_{j=0}^{t-1} q_j v^j = \sum_{\ell=0}^{t-1} (v - k + 4\ell). \]

Then

\[ z_1 + \sum_{j=1}^{t-1} q_j s_j D(\xi_j z_j) \]

\[ + e \sum_{j=1}^{t-1} q_j s_j C(t-1,t)\xi_{t-1}(D(j,t-1)z_j) = 0. \]

Suppose that \( k \) is even, and let

\[ R(v) = v^t + \sum_{j=1}^{t-1} r_j v^j = \sum_{\ell=0}^{t-2} (v - k + 2 + 4\ell). \]

Then

\[ z_1 + \frac{1}{2} \sum_{j=1}^{t-1} (r_j + s_j) D(\xi_j z_j) \]

\[ + \frac{1}{2} e \sum_{j=1}^{t-1} (r_j - s_j) C(t-2,t)\xi_{t-2}(D(j,t-2)z_j) = 0. \]

and

\[ (z_{t-1} - ez_{t-1}) \]

\[ + e \sum_{j=1}^{t-1} s_j \xi_{t-1}(D(j-1,t-1)\xi_{t-1}(z_{t-1} - ez_{t-1})) = 0. \]

There is a generalized Rogers–Ramanujan identity, due to Gordon (14), Andrews (15–17), or Bressoud (18, 19), whose product side coincides with \( \chi(\Omega_v) \) for V the most general standard A^{(1)}-module of positive level (3, 4; cf. ref. 6). (Bressoud’s unified proof (18) of these identities does indeed include the case \( k = 1 \) excluded in ref. 6.) Combining these identities with the infinite systems of new identities for which Eqs. 5, 6, and 7 are the generating functions, we can prove the Conjecture of ref. 6:

Theorem 5. In the notation of Theorem 4, let

\[ k_0 = \min(\lambda(h_0), \lambda(h_1)), \]

and let \( J = 0 \) if \( k \) is even, \( J = 1 \) if \( k \) is odd. Then for all \( n \geq 0 \),

\[ \chi(\Omega_{kn}/\Omega_{kn-1}) = \sum_{\ell \in \mathbb{Z}} C_n \ell^4, \]

where \( C_n \) denotes the number of partitions \( d_1 + \cdots + d_\ell \) of \( \ell \) such that \( 0 < d_i \leq d_{i+1} \leq 2 + d_{i+2} \); if \( d_{i+2} \leq d_i + 1 \), then \( d_i + \cdots + d_{i+2} = k_0 \mod 2 \); and at most \( k_0 \) of the \( d_i \) = 1. In particular, for V coincides with the generalized Rogers–Ramanujan identity of Gordon, Andrews, or Bressoud.

In particular, we have a new proof of the main result (Theorem 1) of ref. 6, our interpretation of the classical Rogers–Ramanujan identities by the \( \delta \)-filtration of \( \Omega \) for the level 3 standard A^{(1)}-modules. At the same time, we obtain a new explicit construction of A^{(1)} for each standard module V of positive level, via an explicit basis for \( \Omega_v \):

Theorem 6. In the notation of Theorems 4 and 5, let \( v_0 \) be a highest weight vector of V. Then \( \Omega_v \) has basis

\[ \{ Z_{-d_0} \cdots Z_{-d_{l-1}} - Z_{-d_l} \cdot v_0 \}. \]
where \( n \geq 0 \) and the \( d_i \) vary as in Theorem 5. The identities 5, 6, 7 give effective recursive definitions of the action of the generators \( Z_k \) of \( \mathcal{I} \) on \( \Omega \), with respect to this basis, and Propositions 2 and 7 give a corresponding explicit construction of \( A_1 \).

Remark: The case \( k = 2 \) of the present theory is much simpler than the cases of larger \( k \) because for \( k = 2, M_2N_2 \) in Theorem 4 is the constant 2, and our algebra \( \mathcal{I} \) is just an infinite-dimensional Clifford algebra. The vacuum space \( \Omega \) is the Fock space for \( \mathcal{I} \), and the module \( V \) is realized in this case very simply as the tensor product of a symmetric Fock space, on which \( \delta \) acts through creation and annihilation operators, with an antisymmetric Fock space, on which the generators of \( \mathcal{I} \) act through creation and annihilation operators.

Remark: The similarity of the polynomials in Theorem 4 with "conical polynomials" (20, 21) is tantalizing.

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