The restricted simple Lie algebras are of classical or Cartan type

(Lie p-algebras)

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Communicated by Nathan Jacobson, April 20, 1984

ABSTRACT The classification of the finite-dimensional restricted simple Lie algebras over an algebraically closed field of characteristic p > 7 is announced. Such an algebra is either of classical type (an analogue over F of a finite-dimensional simple Lie algebra over the complex numbers) or of Cartan type (an analogue over F of one of the infinite Lie algebras of Cartan over the complex numbers).

This paper announces the classification of the finite-dimensional restricted simple Lie algebras over an algebraically closed field of characteristic p > 7. Our theorem states that these algebras are all of classical or Cartan type (the definitions are recalled below). This proves the conjecture of Kostrikin and Safarević (1) made in 1966.

Work on simple Lie algebras of prime characteristic began nearly 50 years ago. Much of this work has concentrated on the case of restricted Lie algebras (also called Lie p-algebras). We recall that a Lie algebra of prime characteristic p is called restricted [a notion introduced by Jacobson (2)] if it has an extra operation x → x^p satisfying certain hypotheses, in particular, (x y)^p = x^p y^p (3). Many important examples of Lie algebras are restricted. In particular, the algebra Der A of derivations of any (nonassociative) algebra A is restricted (where the pth power of a derivation is that in End A). In restricted Lie algebras one has the use of certain technical tools [e.g., the Jordan–Chevalley–Safarevic decomposition of an element into its semisimple and nilpotent parts (4)] not available for arbitrary Lie algebras. If L is any simple Lie algebra of prime characteristic p, then L may be identified with a subalgebra of Der L and L ⊆ L ⊆ Der L, where L is the smallest restricted subalgebra of Der L containing L.

The techniques which yield the classification of the finite-dimensional simple Lie algebras over the complex numbers C (by reduction to the classification of irreducible root systems) make heavy use of Lie’s Theorem and the nondegeneracy of the Killing form. These techniques fail over fields of prime characteristic since Lie’s Theorem does not hold and the Killing form may be identically zero.

We shall now describe the restricted Lie algebras of classical and Cartan type (Section 1) and then state our result and briefly describe some of the main steps in the proof (Section 2). The complete proof, which is very long, will be given elsewhere.

1. Algebras of Classical and Cartan Type

From now on F will denote an algebraically closed field of characteristic p > 7; we will state all results for algebras over F.

Let L be a finite-dimensional simple Lie algebra over C, L_{p} the Z-span of a Chevalley basis in L, and L_{p} = L_{p} ⊕_{p} F. Then z(L_{p}), the center of L_{p}, has dimension ≤1 and L_{p}/z(L_{p}) is restricted and simple. The algebras which arise in this way are said to be of classical type. Thus, the simple Lie algebras of classical type over F correspond to the irreducible root systems: A_{n} (n ≥ 1), B_{n} (n ≥ 3), C_{n} (n ≥ 2), D_{n} (n ≥ 4), E_{6}, E_{7}, E_{8}, F_{4}, and G_{2}. The Lie algebras of classical type have been characterized axiomatically by Mills and Seligman (5). They include all simple Lie algebras with nondegenerate Killing form.

Let B_{n} denote the commutative associative algebra of p-truncated polynomials, with generators x_{1}, ..., x_{n} and relations x_{i}^{p} = 0 for i = 1, ..., n, and let W_{n} = Der B_{n}. Then W_{n} is a restricted simple Lie algebra of dimension n p^{n} called a Jacobson–Witt algebra [Chang (6), Zassenhaus (7), Jacobson (8)]; each element of W_{n} is uniquely expressible as Σ_{i=1}^{n} f_{i}(t_{i}) (t_{i} ∈ B_{n}), where D_{i} ∈ W_{n} is defined by D_{i}(x_{j}) = δ_{ij} for i, j = 1, ..., n.

We now recall the notation for certain elements of the exterior algebra (over B_{n}) of differential forms in dx_{1}, ..., dx_{n}. Let

\[ \omega_{S} = dx_{1} \wedge ... \wedge dx_{n}, \]
\[ \omega_{H} = \sum_{i=1}^{r} dx_{i} \wedge dx_{i+r}, \quad n = 2r, \]

and

\[ \omega_{K} = dx_{2r+1} + \sum_{i=1}^{r} x_{i+r} dx_{i} - x_{i} dx_{i+r}, \quad n = 2r + 1. \]

Define

\[ S_{n} = \{ D ∈ W_{n}|D \omega_{S} = 0\}, \]
\[ H_{n} = \{ D ∈ W_{n}|D \omega_{H} = 0\}, \quad n = 2r, \]

and

\[ K_{n} = \{ D ∈ W_{n}|D \omega_{K} ∈ B_{n}\omega_{S}, \quad n = 2r + 1. \]

[Here the action of D on a differential form ω is given by D(η + τ) = D(η) + D(τ), D(f) = (Df)x + f(Dη), D(η ∧ τ) = (Dη) ∧ τ + η ∧ (Dτ), and D(df) = D(DF), where df = Σ (Df)dx_{i}.] Let A^{m,n} denote the m-th derived algebra of the Lie algebra A. The algebra S_{n}^{[1]}(p ≥ 3, is a restricted simple Lie algebra of dimension (n − 1)(p^{n} − 1) [discovered by Frank (9)]. The algebra H_{n}^{[2]}(p ≥ 2, is a restricted simple Lie algebra of dimension p^{n} − 2 [discovered by Albert and Frank (10)]. The algebra K_{n}^{[1]}(p ≥ 2, is a restricted simple Lie algebra of dimension p^{n} if n + 3 ≠ 0 (mod p) and of dimension p^{n} − 1 if n + 3 = 0 (mod p) [discovered, in another form, by Frank (11)].

Kostrikin and Šafareviš (1) observed that for each of the four families of simple infinite Lie algebras over C of Cartan (12–14) there is an analogous family of restricted simple Lie algebras over F. Three of these were the known families W, S, and H. The fourth was later shown by Celousov (15) (as had been conjectured by Kostrikin and Šafareviš) to be identical to the family of algebras K_{n}, n = 2r + 1 ≥ 3, discovered...
by Frank. Kostrikin and Šafarevič called these algebras the restricted simple Lie algebras of Cartan type over $F$.

2. The Classification Theorem

After making these observations, Kostrikin and Šafarevič conjectured that all finite-dimensional restricted simple Lie algebras over $F$ are of classical or Cartan type. For restricted simple Lie algebras with a one-dimensional Cartan subalgebra this conjecture agreed with Kaplansky’s theorem (16) that such an algebra is isomorphic to $sl(2)$ or $W_1$. For restricted simple Lie algebras with a two-dimensional Cartan subalgebra this conjecture has been proved by the present authors (17). For a survey of known classification results see ref. 18.

Our main result is that this conjecture is true.

THEOREM 1. Let $F$ be an algebraically closed field of characteristic $p > 1$. Let $L$ be a finite-dimensional restricted simple Lie algebra over $F$. Then $L$ is of classical or Cartan type.

Restricted simple Lie algebras which are not of classical or Cartan type are known to exist for fields of characteristics 2, 3, and 5. We know of no such examples for characteristic 7.

Nonrestricted simple analogues over $F$ (denoted $W(m:n)$ and $X(m:n;\Phi)^{(2)}$ for $X = S, H, and K$) of the infinite Lie algebras of Cartan type over $\mathbb{C}$ are also known (17, 19, 20). These algebras are also known as algebras of Cartan type. They include all known nonclassical finite-dimensional simple Lie algebras over $F$.

The proof of Theorem 1 has two main parts, which generalize, respectively, the determination (17) of the restricted semisimple Lie algebras with a two-dimensional Cartan subalgebra and the classification (21) of the restricted simple Lie algebras with a Cartan subalgebra which is a torus. (Recall that a torus is by definition a subalgebra, necessarily abelian, which consists entirely of semisimple elements.) The first part is devoted to the classification of certain restricted semisimple Lie algebras containing a two-dimensional maximal torus, while the second part uses this information to obtain the general classification.

To explain in more detail we need to consider certain filtrations of $L$. Let $L_0$ be a maximal subalgebra of $L$ and $L_{-1} \supseteq L_0$ be an $L_0$-submodule such that $L_{-1}/L_0$ is an irreducible $L_0$-module. Following a refinement by Weisfeiler (22) of a classical technique used in characteristic zero (cf. ref. 14) we define a filtration of $L$ by

$$L_{i-1} = \{L_{-1}, L_i\} + L_i$$

and

$$L_{i+1} = \{x \in L_i | [x, L_{-1}] \subseteq L_0\}$$

for $i < 0$ and $i > 0$.

Let $G = \Sigma G_i$, with $G_i = L_i/L_{i+1}$, be the associated graded algebra. Also let $G'$ denote the subalgebra of $G$ generated by $\Sigma_{i \in \mathbb{N}} G_i$ and $N(G)$ denote the unique maximal ideal of $G$ contained in $\Sigma_{i \in \mathbb{N}} G_i$. One attempts (as in the characteristic zero case) to determine first $G$ and then $L$. In prime characteristic this program was begun by Kostrikin and Šafarevič (1, 23). For restricted algebras one has the following result of Kac (19).

RECOGNITION THEOREM. Let $L$ be a finite-dimensional restricted simple Lie algebra over $F$. Let $L$ be filtered as above. Suppose that $G_0$ is a direct sum of restricted ideals each of which is either
(a) classical simple,
(b) $sl(n), pg(n)$, or $gl(n)$, where $p, n$, or $c$ abelian.
Suppose further that the representation of $G_0$ on $G_1$ is faithful and that $N(G) = (0)$. Then $L$ is of classical or Cartan type.

In practice the application of this theorem is difficult, since one does not know a priori that a restricted simple Lie algebra has an $L_0$ for which the hypotheses on $G_0$ are satisfied. Our proof consists of showing the existence of such an $L_0$.

Our construction of $L_0$ depends upon the notion of a section of a Lie algebra. Suppose $R$ is a maximal torus of a restricted Lie algebra $A$. Then $A$ has the decomposition

$$A = \sum_{\gamma \in \Gamma} A_\gamma,$$

where $\Gamma$ consists of those $\gamma$ in the dual space $R^*$ of $R$ for which the space

$$A_\gamma = \{a \in A | [r, a] = \gamma(r) a\ \forall r \in R\}$$

is nonzero. Let $\Delta$ denote the $Z$-span, $Z\Gamma$, (in $R^*$) of $\Gamma$. If $X \subseteq \Delta$ we define

$$A^{(X)} = \sum_{\gamma \in Z\Gamma} A_\gamma$$

and

$$A[X] = A^{(X)}/\text{solvable}(A^{(X)})$$

[where solvable($B$) denotes the solvable radical of $B$.] We call $A[X]$ a section of $A$.

We say that a maximal torus $R \subseteq A$ is standard if $z_2(R)$, the centralizer of $R$ in $A$ (which is also a Cartan subalgebra of $A$), contains a nil ideal $I$ such that $z_4(R) = R + I$. It is known (24) that if $L$ is a restricted simple Lie algebra, then any maximal torus in $L$ is standard. It follows that if $A$ is any section of a restricted simple Lie algebra, then any maximal torus in $A$ is standard.

The first step in our proof is to determine all finite-dimensional restricted semisimple Lie algebras over $F$ which contain a one-dimensional maximal torus which is standard and to classify the standard tori in such algebras.

PROPOSITION 2. Let $A$ be a finite-dimensional restricted semisimple Lie algebra over $F$ containing a one-dimensional maximal torus $R$ which is standard. Then $A$ is isomorphic to one of the (eight) algebras in the following list by an isomorphism which takes $R$ to the span of the indicated element.

\[
\begin{pmatrix}
 sl(2), & (1, 0, 0, -1)
\end{pmatrix};
\]

\[(W_1, x_1 D_1);\]

\[(W_1, x_1 + 1 D_1);\]

\[(H^{(2)}_2 + D, \Psi(x_1 D_1 - x_1 D_2), \text{ where } D \text{ is the span of one of the following sets: } \{x_1^{P+1} D_2, x_1^{P+1} x_1^{P-1} D_1 - x_1^{P} x_1^{P-1} D_2\},\]

\[(x_1^{P+1} D_2, x_1^{P+1} x_1^{P-1} D_1 - x_1^{P} x_1^{P-1} D_2),\]

\[(x_1^{P+1} D_2, x_1^{P+1} x_1^{P-1} D_1 - x_1^{P} x_1^{P-1} D_2),\]

\[\text{and where } \Psi \text{ is an automorphism of } H_2;\]

\[(H^{(2)}_2 + D, \Psi(x_1 + 1 D_1 - x_2 D_2), \text{ where } D \text{ and } \Psi \text{ are as above};\]

\[(H^{(2)}_2 + D, \Psi(x_1 + 1 D_1 - x_2 D_2), \text{ where } D \text{ and } \Psi \text{ are as above};\]

\[\text{and where } \Psi \text{ is an automorphism of } H_2;\]

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Now suppose that $A$ has standard maximal torus $R$, so that $A_0 = z_q(R) = R + I$, where $I$ is a nil ideal in $z_q(R)$. Define

\[ K_0(A) = \{ x \in A_0 | \alpha(x, A_{-\alpha}(x)) = 0 \} \]

\[ N_0(A) = \{ x \in K_0(A) | \ker (x, A_{-\alpha}(A)) \subseteq I \}, \]

and

\[ T_0(A) = \{ x \in A_0 | \ker (x, A_{-\alpha}(x)) \subseteq I \}. \]

Note that $K_0(A) = N_0(A) = T_0(A) = A_0$. It is clear that

\[ T(A) = \sum_{a \in \Delta} T_0(A) \]

is a subalgebra of $A$.

Define $\theta \neq \alpha \in \Delta$ to be proper if $L_{\theta a}(A) = K_{\theta a}(A)$ for some $i, 1 \leq i \leq p - 1$, and 0 to be improper otherwise. Let $P = \{ \alpha \in \Delta | \alpha \text{ is proper} \}$ and $n(R) = |P|/(p - 1)$. We say that $R$ is an optimal maximal torus of $A$ if

(i) $R$ is a torus of maximal dimension in $A$,

(ii) $R$ is standard,

and

(iii) $S$ is any torus of $A$ satisfying $i$ and $ii$, then $n(S) = n(R)$.

By explicit study of the Cartan decompositions of the algebras enumerated in Proposition 2 we obtain information on rank one sections.

**Proposition 3.** Let $A$ be a restricted Lie algebra over $F$ and $\alpha$ a root with respect to a standard maximal torus $R$. Then:

(a) If $\alpha$ is proper one of the following occurs:

(i) $A[\alpha] = (0)$ and $A_{\alpha} = K_{\alpha}(A)$ for all $i$.

(ii) $A[\alpha] = s(2)$ and there is some $i, 1 \leq i \leq p - 1$, such that $A_{\alpha} = K_{\alpha}(A)$ for all $j \neq \pm i$ and $\dim A_{z_{\pm i}}/K_{z_{\pm i}}(A) = 1$.

(iii) $A[\alpha] = W_1$, there is a homomorphism of $A^{\alpha}$ onto $W_1$ which maps $R$ to $F x_1 D_1$, and there is some $i, 1 \leq i \leq p - 1$, such that $A_{\alpha} = K_{\alpha}(A)$ for all $j \neq \pm i$ and $\dim A_{z_{\pm i}}/K_{z_{\pm i}}(A) = 1$.

(b) If $\alpha$ is improper one of the following occurs:

(i) $A[\alpha] = W_1$, there is a homomorphism of $A^{\alpha}$ onto $W_1$ which maps $R$ to $F(x_1 + 1)D_1$ and $\dim (A_{\alpha}/K_{\alpha}(A)) = 1$ for all $i, 1 \leq i \leq p - 1$.

(ii) $A[\alpha] = B$ where $B$ is $H_{i}^{2}$ or $H_{i}^{2} + D$, there is a homomorphism of $A^{\alpha}$ onto $B$ which maps $R$ to $F(y_{1}D_1 - x_2 D_2)$, $\Psi \in Aut B^{(3)}$, and there is some $i, 1 \leq i \leq p - 1$, such that $A_{\alpha} = K_{\alpha}(A)$ for all $j \neq \pm i, \pm 2i, \dim A_{z_{\pm i}}/K_{z_{\pm i}}(A) = 1$ and $\dim A_{z_{\pm 2i}}/K_{z_{\pm 2i}}(A) = 1$.

(c) $S \subseteq A \subseteq Der S$, where $S$ is one of $W_1(2), H_{2}(2.1)$, $H_{2}(2.2)$, $H_{2}(2.1)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$, $H_{2}(2.2)$, $F_{x_{1}D_{1}} H_{2}(2.1)$.

Now let $A$ be an arbitrary finite-dimensional restricted Lie algebra such that any torus of maximal dimension is standard. A technique [due to Winter (25)] exists for truncating exponentials

\[ \sum_{i=0}^{p-1} (ad x)^{i}/i! \]

of root vectors to pass from one torus of maximal dimension in $A$ to another. It is possible to use this technique to make any one root $\alpha$ proper. This is not, a priori, sufficient to allow us to make all roots proper, since in making $\alpha$ proper we might make a proper root $\beta$ improper. If this were to occur, the same phenomenon would occur in the rank two section $A[\alpha, \beta]$. By studying the possible rank two sections (described in Proposition 5) we can show that this does not occur. This gives the following result.

**Lemma 6.** Let $L$ be a restricted Lie algebra over $F$. Then there is a torus, $R$, of maximal dimension, with respect to which all roots of $L$ are proper.

For any restricted Lie algebras $M \supseteq N$ let $\mathfrak{e}(M, N)$ denote the set of all (ad $N$)-invariant subalgebras of $M$ for which every composition factor is abelian or classical simple. Study of the algebras listed in Proposition 2 allows us to prove the following result.

**Lemma 7.** Let $A$ be a restricted Lie algebra over $F$, $R$ be a standard maximal torus in $A$, and $\alpha$ a proper root of $A$ with respect to $R$. Then $\mathfrak{e}(A^{\alpha}, R)$ contains a unique element of maximal dimension.

Now let $L$ be restricted simple and $R$ be a maximal torus of $L$ with respect to which all roots of $L$ are proper. Let $Q^{(\alpha)}$ denote the element of maximal dimension in $\mathfrak{e}(L, R)$ given by Lemma 7. We write $Q^{(\alpha)} = \sum_{\alpha \in \Delta} Q_{\alpha}$ and define $Q = \sum_{\alpha \in \Delta} Q_{\alpha}$. The following result shows that $Q$ is a large subalgebra of $L$.

**Proposition 8.** $Q$ is a subalgebra of $L$, $\dim (L_{Q}/Q_{Q}) \leq 1$, and if $\alpha, \beta \in \Delta$ then $|\{ \alpha \in \alpha \in \Delta | B_{\alpha \beta} = 0 \} | \leq 2$.

This is a result which can be proved in the rank two sections. Unfortunately, it is not true for all of the algebras described in Proposition 5. However, we are able to show that not all of the algebras of Proposition 5 can occur as sections of simple algebras and that Proposition 8 does hold for those which can occur, thus proving the proposition.

If $Q = L$, we can show that $L$ is classical. If $Q \neq L$ we take $L_{Q}$ to be a maximal subalgebra of $L$ containing $Q$ and show that $L_{Q}$ satisfies the hypotheses of the recognition theorem, thus proving Theorem 1.
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