Algebraic K-theory of spaces stratified fibered over
hyperbolic orbifolds

(pseudo-isotopy/sectional curvature/discrete subgroup/finite group/geodesic flow)

F. T. FARRELL† and L. E. JONES‡

†Department of Mathematics, Columbia University, New York, NY 10027; and ‡Department of Mathematics, State University of New York, Stony Brook, NY 11794

Communicated by Hyman Bass, March 28, 1986

ABSTRACT Among other results, we rationalize calculate
the algebraic K-theory of any discrete cocompact subgroup
of a Lie group G, where G is either O(n, 1), U(n, 1), Sp(n, 1), or Fq, in terms of the homology
of the double coset space \( \Gamma \setminus \Gamma G/K \), where \( K \) is a maximal cocompact subgroup of G.
We obtain the formula
\[ K_m(\Gamma) \otimes \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Z}G}(\Gamma \setminus \Gamma G/K; \mathcal{X}_t) \],
where \( \mathcal{X}_t \) is a stratified system of \( Q \) vector spaces over \( \Gamma \setminus \Gamma G/K \) and the vector space \( \mathcal{X}_t \) corresponding to the
double coset \( \Gamma \setminus \Gamma G/K \) is isomorphic to \( K_m(\Gamma \setminus \Gamma G/K; \mathbb{Q}) \).
Let \( \Gamma' \) be a stratified system of \( G \) vector spaces over \( \Gamma \setminus \Gamma G/K \).
Note \( \Gamma' \setminus \Gamma G/K \) is a finite subgroup of \( \Gamma \). Earlier, a similar
formula for discrete cocompact subgroups \( \Gamma \) of the group of
rigid motions of Euclidean space was conjectured by F. T.
Farrell and W. C. Hsiang and proven by F. Quinn.

Let \( M \) be a closed (i.e., compact and without boundary) con-
nected Riemannian manifold all of whose sectional curva-
tures are strictly negative and

\[ N^1 \subset N^2 \subset N^3 \subset \ldots \] [1]

be a sequence of connected compact smooth manifolds. Let \( F \) be a finite group together with actions of \( F \) on \( M \) via isometries
and on each \( N^i \) via smooth maps; the action of \( F \) on each \( N^i \) is free but the action on \( M \) need not be free. We assume the smooth embedding \( N^i \subset N^{i+1} \) is both \( F \)-equivariant
and \( i \)-connected. Let \( N^\infty = \lim_{i \to \infty} N^i \) and give \( N^\infty \) the
direct limit topology; the induced action of \( F \) on \( N^\infty \) is free.
Let \( E^i \) (i = 1, 2, \ldots, \infty) denote the orbit space of \( M \times N^i \)
under the diagonal action of \( F \); note \( E^\infty \) is direct limit of \( E^i \).
Let \( X \) be the orbit space \( M/F \) and \( \rho_1: E^i \to X \) (i = 1, 2, \ldots, \infty) the map induced from the canonical projection of \( M \times N^i \) onto \( M \). Each \( \rho_i \) is a stratified system of fibrations
on \( X \) (cf. definition 8.2 of ref. 1). Abbreviate the notation \( \rho \), \( E^\infty \), \( N^\infty \) to \( \rho \), \( E \), \( X \), \( N \); respectively.

For any pair of closed geodesics \( \alpha_1, \alpha_2 \subset M \) (considered
only as point sets, without orientation or parameterization),
set \( \alpha_1 \sim \alpha_2 \) if \( g \alpha_1 = \alpha_2 \) for some \( g \in F \). Let \( F_\alpha \) denote the
subgroup all \( g \in F \) such that \( g \) leaves the closed geodesic \( \omega \)
invariant and restricted to \( \omega \) is orientation preserving. Let \( F_\omega \) be the normal subgroup of \( F_\alpha \) that pointwise fixes \( \omega \). Note the factor group \( F_\alpha F_\omega / F_\omega \) is cyclic. Pick an action of \( F_\alpha \) on \( S^1 \) the
circle of radius one—via orientation preserving isometries
so that \( F_\alpha \) acts trivially and \( F_\alpha F_\omega / F_\omega \) freely. Apply the
construction of the paragraph above with \( M \), \( F \) replaced by
\( S^1 \), \( F_\alpha \) (respectively) and denote by \( \rho_\alpha: E_\alpha \to S^1 \) (in place of
\( \rho: E \to X \)) the corresponding stratified system of fibrations.

Let \( p(Y) \) denote the stable topological pseudo-isotopy \( \Omega \)-
spectrum associated to a topological space \( Y \) (cf. ref. 2 and
section 5 of ref. 1). To a stratified system of fibrations \( \rho: E \to X \) (as above), Quinn (sections 5 and 8 of ref. 1) associates
a homology \( \Omega \)-spectrum \( H(X; \rho(p)) \) with “twisted” spectrum coefficients and a map of \( \Omega \)-spectra

\[ H(X; \rho(p)) \to H(E). \] [2]

Denote its cofiber (in the category of spectra) by \( Np(X; \rho) \);
then \( Np(X; \rho) \) is also an \( \Omega \)-spectrum.

Let \( \alpha_1, \alpha_2, \ldots \) be a sequence of closed geodesics in \( M \)
such that each \( \sim \) equivalence class is represented exactly
once in the sequence. Let \( X_\infty = Np(S^1; \rho_\omega) \) denote

\[ \text{direct limit } X_\infty = Np(S^1; \rho_\omega). \] [3]

Theorem 1. For each of the stratified systems of fibrations \( \rho: E \to X \) constructed in the first paragraph of this announcement, there are two homotopy equivalences \( \approx \) of \( \Omega \)-spectra

\[ \begin{align*}
(i) \quad \rho(E) & \approx H(X; \rho(p)) \times Np(X; \rho) \\
(ii) \quad Np(X; \rho) & \approx X_\infty = Np(S^1; \rho_\omega).
\end{align*} \]

Part of the usefulness of Theorem 1 rests on an Atiyah–
Hirzebruch type spectral sequence, established by Quinn (cf. theorem 8.7 of ref. 1) with

\[ E^2_{p,q} = H_q(X; \pi_p(p(\rho))), \] [4]

which abuts to \( \pi_{p+q}(H(X; \rho(p))) \), where \( \pi_p(p(\rho)) \) denotes the stratified system of abelian groups over \( X \) with the group
\( \pi_q(p(\rho(\omega)(x))) \) corresponding to the point \( x \in X \).

Remak 1: When \( \alpha \) is a stratified system of abelian groups
over \( X \) (cf. section 8 of ref. 1), the homology groups \( H_\alpha(X; \alpha) \)
can be calculated as follows. Let \( T \) be a finite triangulation of
\( X \) such that each strata is a subcomplex; then \( \alpha \)
restricted to each open simplex is a constant system of coefficients
and \( H_\alpha(X; \alpha) \) is the homology of a chain complex \( \alpha \)
with

\[ C_n = \bigoplus_{\alpha \in T_n} \alpha(\delta), \] [5]

where \( T_n \) is the set of \( n \)-simplices in \( T \) and \( \delta \) denotes the
barycenter of \( \alpha \).

Remark 2: The \( E^2_{p,q} \) terms in spectral sequence 4 can be expressed in the following alternative manner. Let \( \Gamma \) be the
extension of \( \pi_1M \) determined by the action of \( F \) on \( M \) (ex-
\implicitly, \( \Gamma \) is isomorphic to the factor group \( \pi_1M / \pi_1\mathbb{Z}N \) and
\( \mathbb{Z}(\Gamma) \) denote the category with objects the finite subgroups
of \( \Gamma \). Each element \( y \in F \) determines a morphism \( G_1 \to G_2 \)
provided \( yG_1y^{-1} \subseteq G_2 \). Then, \( E^2_{p,q} \) is isomorphic to the direct

\[ \lim_{G \to \Gamma} \pi_q(p(N/p(G))), \] [6]

The publication costs of this article were defrayed in part by page charge
payment. This article must therefore be hereby marked "advertisement"
in accordance with 18 U.S.C. §1734 solely to indicate this fact.
where \( p: \Gamma \to F \) is the canonical projection and \( N/\rho(G) \) is the orbit space. (See ref. 3 for a discussion of the contravariant form of this type of limit.) A basic result of E. Cartan is used in demonstrating this isomorphism; namely, a finite group of isometries of a complete simply connected Riemannian manifold all of whose sectional curvatures are nonpositive has a fixed point (cf. theorem 13.5 of ref. 4). A similar observation has been made by Quinn (lemma 1.4 of ref. 5) for the case in which \( \Gamma \) is a poly \( Z \) by finite group.

**Remark 3:** Theorem 1 implies

\[
\mathcal{A}(E) = H(X; \mathcal{A}(p)) \times Np(X; p),
\]

where \( \mathcal{A}(\_ ) \) denotes the \( \Omega \)-spectrum associated either to Waldhausen’s algebraic K-theory of spaces functor \( Y \to A(Y) \) or to the stable smooth pseudo-isotopy of spaces functor \( Y \to \mathcal{P} \mathcal{B}(Y) \).

The following result makes Theorem 1 effective for the rational calculation of many algebraic K-groups.

**Theorem 2.** If \( p: E \to X \) is a stratified system of fibrations as in Theorem 1 and \( N \) is aspherical with \( \text{Wh}_n(\pi_1 N \times S^1) \otimes \mathbb{Q} = 0 \) for all integers \( n \), then \( \text{Wh}_n(X; p) \otimes \mathbb{Q} = 0 \).

**COROLLARY 1.** Assuming that \( N \) is contractible and denoting \( \pi_1 E \) by \( \Gamma \), we have the following calculations valid for all integers \( n \):

\[
\begin{align*}
\text{(i) } \text{Wh}_n(\Gamma) \otimes \mathbb{Q} &= \bigoplus_{i=0}^{\infty} H_i(X; \text{Wh}_n-(G_i) \otimes \mathbb{Q}) \\
\text{(ii) } K_n(\Gamma) \otimes \mathbb{Q} &= \bigoplus_{i=0}^{\infty} H_i(X; K_n-(G_i) \otimes \mathbb{Q}),
\end{align*}
\]

where \( G_y = \pi_1(p^{-1}(y)) \) for \( y \in X \) and \( \text{Wh}_n-(G_i) \otimes \mathbb{Q}, \ K_n-(G_i) \otimes \mathbb{Q} \) are the corresponding stratified systems of abelian groups over \( X \).

A formula, similar to those in Corollary 1, for discrete cocompact subgroups \( \Gamma \) of the group of rigid motions of Euclidean space has been conjectured by Farrell and Hsiang (6) and proven by Quinn (5).

**Remark 4:** In Corollary 1, \( H_2(X; \text{Wh}_n-(G_i) \otimes \mathbb{Q}) \) and \( H_2(X; K_n-(G_i) \otimes \mathbb{Q}) \) can be identified with

\[
\lim_{G \in \mathcal{G}(\Gamma)} \text{Wh}_n(G) \otimes \mathbb{Q} \quad \text{and} \quad \lim_{G \in \mathcal{G}(\Gamma)} K_n(G) \otimes \mathbb{Q};
\]

respectively.

**Theorems 1 and 2** yield the following extension of corollaries 1 and 2 of ref. 7.

**COROLLARY 2.** Let \( M_1, M_2, \ldots, M_m \) be closed connected Riemannian manifolds all of whose sectional curvatures are strictly negative and let \( C \) be a finitely generated free abelian group; then

\[
\text{Wh}_n(\pi_1 M_1 \oplus \pi_1 M_2 \oplus \cdots \oplus \pi_1 M_m \oplus C) \otimes \mathbb{Q} = 0
\]

for all integers \( n \).

**Definition 1:** A group \( \Gamma \) is strongly virtually negatively curved if \( \Gamma \) is isomorphic to \( \pi_1 E \), where \( E \) is constructed by the method described in the first paragraph of this announcement with the additional constraint that \( N \) is contractible.

**Remark 5:** Any discrete cocompact subgroup of a Lie group \( G \), where \( G \) is either \( O(n, 1), U(n, 1), Sp(n, 1), \) or \( F_2 \), is strongly virtually negatively curved. More generally, any group, which contains a subgroup of finite index isomorphic to the fundamental group of a compact connected local symmetric space all of whose sectional curvatures are strictly negative, is strongly virtually negatively curved.

The second sentence of Remark 5 is a consequence of Mostow’s rigidity theorem (8) together with the positive solution of Nielsen’s problem (9, 10).

When \( \Gamma \) is strongly virtually negatively curved, the groups \( G_y = \pi_1(p^{-1}(y)) \) occurring in Corollary 1 are finite because they are isomorphic to subgroups of \( F \). Hence, we can use the extensive knowledge of the algebraic K-theory of finite groups due to Swan (11), Bass (12), Bass and Murthy (13), Quillen (14), Borel (15), Carter (16), and others to analyze the algebraic K-theory of \( \Gamma \) by Corollary 1 and Theorem 1 (for lower K-theory). For instance, we have the following result.

**COROLLARY 3.** If \( \Gamma \) is strongly virtually negatively curved (cf. Definition 1 and Remark 5), then

(i) \( K_n(\Gamma) = 0 \) for all \( n < -1 \);
(ii) \( K_{-1}(\Gamma) \) is a finitely generated abelian group;
(iii) \( K_{-1}(\Gamma) \) is generated by the images of \( K_{-1}(G) \), as \( G \) varies over the finite subgroups of \( \Gamma \), under the map functorially induced by the inclusion of \( G \) into \( \Gamma \).
(iv) \( \sigma; : K_n(\Gamma) \otimes \mathbb{Q} \to K_n(\Gamma) \otimes \mathbb{Q} \) is the zero homomorphism where \( \sigma; : \Gamma \to \Gamma \) is inclusion;
(v) \( K_n(\Gamma) \otimes \mathbb{Q} \) is a finite-dimensional \( \mathbb{Q} \) vector space for all integers \( n \).

**Remark 6:** Corollary 3 is perhaps true for a much larger class of groups \( \Gamma \); in particular, we conjecture it for all finitely generated subgroups \( \Gamma \) of \( GL_n(\mathbb{C}) \).

**Remark 7:** Property iv of Corollary 3 is an analogue of Swan’s result that \( P \otimes \mathbb{Q} \) is a free \( \mathbb{Q} \)-module whenever \( P \) is a finitely generated projective \( \mathbb{Z} \)-module and \( \mathbb{F} \) is a finite group. In fact, Swan’s result is used in the proof of property iv.

**Remark 8:** More precise information concerning \( \dim_{\mathbb{Q}} K_n(\Gamma) \otimes \mathbb{Q} \) should be obtainable from Corollary 1 by use of refs. 11, 12, 15, and 16.

**Remark 9:** Properties ii and iii can be made more specific by the following:

\[
K_{-1}(\Gamma) = \lim_{G \in \mathcal{G}(\Gamma)} K_{-1}(G).
\]

We end this announcement with a brief indication of proof. For instance, Theorem 2 is obtained by a Frobenius induction technique similar to that used in refs. 6 and 17. Corollaries 1 and 3 follow from Theorems 1 and 2 using \( \Omega \)-spectra and maps of \( \Omega \)-spectra when tensored with \( \mathbb{Q} \) are directly analyzable in terms of Eilenberg–MacLane spectra; the details are nontrivial.

Our main result, Theorem 1, is proved by an \( F \)-equivariant form of the argument we used to prove theorems 1 and 2 of ref. 7. Our proof again makes crucial use of Anosov’s work (18) about the dynamics of the geodesic flow on the unit sphere bundle of a closed Riemannian manifold all of whose sectional curvatures are strictly negative. Moreover, we make important use of Quinn’s work (1), which we did not use in ref. 7. There, the work contained in refs. 19–22 was used instead.

To prove Theorem 1 and Remark 3, we must analyze spectra by using standard algebraic topology methods together with ideas used in the proofs of the fundamental theorem of algebraic K-theory (23) and of its twisted version (24).

This work was supported in part by grants from the National Science Foundation.