Symbolic method in invariant theory

(Young symmetrizers/straightening algorithms/skew-symmetric tensors/standard bases/superalgebras)

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ABSTRACT A symbolic method based on an extension of the straightening algorithm is developed for the representation of joint invariants of symmetric and skew-symmetric tensors. For skew-symmetric tensors, the method holds over infinite fields of arbitrary characteristic.

A rigorous foundation of the symbolic method for the representation of invariants of binary forms, used by Clebsch (1) and Gordan (2), has been given by Kung and Rota (3). An extension of the method to skew-symmetric tensors was attempted by Weitzenböck (4, 5). We develop a symbolic method for the representation of joint invariants of symmetric and skew-symmetric tensors. For skew-symmetric tensors, the present symbolic representation holds over infinite fields of arbitrary characteristic. The method is based on an extension of the straightening algorithm (6–9) to algebras containing positively and negatively signed elements.

Section 1. Superalgebras

Let Mon(A) be the free monoid (i.e., semigroup with identity) generated by set A. The identity of Mon(A) is denoted by 1, and the elements of Mon(A) are words. The product of words is juxtaposition. The identity of Mon(A) is the empty word. If w E Mon(A), then w = x1x2,..., xn with xi E A. The i-th entry of the word w is the element xi. Set length(w) = n. For w E Mon(A) and for a E A, let cont(w; a) be the number of entries of w that are equal to a. Let cont(w) = the function a -*+ cont(w; a).

If m is a multisubset of A [i.e., a function a -*+ m(a) E Z+], w has content m when cont(w; a) = m.

A signed set is a set A, together with three mutually disjoint subsets A+, A0, and A- such that A+ U A0 U A- = A, whose elements are called positive, neutral, and negative, respectively.

The extended signed set A2 = A2+ + A20 + A2- of a signed set A is defined by setting A2+ = A6, A20 = A- , and A2- to be the set of all ordered pairs a0 with a E A and i = 1, 2, .... The element a0 is the i-th divided power of a E A; i = a0 = a. The monoid Div(A) = Mon(A2) is the divided powers monoid generated by the signed set A. The monoid homomorphism Disp: Div(A) -*+ Mon(A) is generated by setting Disp(x) = x if x E A+ U A0 and Disp(a0) = a if a E A-. Example: if w = aw0b c0d E Div(A), then Disp(w) = a a a a b c c d = a b c c d E Mon(A). If w E Div(A), set Length(w) = length(Disp(w)), and Cont(w) = cont(Disp(w)).

The parity [w] of w E Div(A) is 0 if

\[ \sum_{a E A} \text{cont(Disp(w); x)} \]

is even, and 1 otherwise.

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A Young diagram D = (w1, w2, ..., wn) on a set A is a sequence of words w1 E Mon(A) such that length(w1) = length(w2) = ... . The integer n is the number of rows of the Young diagram D, and the vector \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \) is the shape of D, where \( \lambda_i = \text{length}(w_i) \).

Set cont(D) = cont(w1) + cont(w2) + ... + cont(wn). Let \( \lambda \) and \( \lambda' \) be shapes. Set \( \lambda \leq \lambda' \) when \( \Sigma \lambda_i = \Sigma \lambda'_i \), and \( \lambda_1 \leq \lambda'_1, \lambda_1 + \lambda_2 \leq \lambda'_1 + \lambda'_2, \ldots \) (dominance partial order).

If \( w_1 = x_1x_2 ... x_{\lambda_1}, w_2 = x_1x_2 ... x_{\lambda_2}, \ldots \) are elements of \( \text{Mon}(A) \), then the words

\[ \tilde{w}_1 = x_1x_2 ... x_{\lambda_1}, \text{ and } \tilde{w}_2 = x_1x_2 ... x_{\lambda_2}, \ldots \]

where \( \tilde{\lambda}_i \) equals the number of words of \( \lambda_i \) of \( D \) such that \( \text{length}(w_i) = j \), defines the dual Young diagram \( D = (\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n) \), where \( k = \tilde{\lambda}_i \) is the number of columns of \( D \) and \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \) is the dual shape. Clearly \( \tilde{D} = D \) and \( \text{cont}(\tilde{D}) = \text{cont}(D) \).

Let A be a signed set. For \( w E \text{Mon}(A) \), define stand(w) to be the unique word in Div(A) such that (i) Disp(stand(w)) = w, (ii) if \( \text{stand}(w) = y_1y_2 ... y_k \), then \( y_i E A_+ \), and if \( y_i = a^{0} \) for some \( a E A_{+} \) and some positive integer i, then \( y_{i-1} \neq a^{0} \) and \( y_{i+1} \neq a^{0} \) for all integers r and s. Example: stand(Disp(a^{0}a^{0}b)) = a^{0}b).

If A is a signed set, then \( \text{Tens}[A] \) be the algebra over \( Z \) generated by \( \text{Div}(A) \) — that is, the module of all finite linear combinations

\[ f = \sum_{i} c_i w_i, \quad c_i, d_j E \text{Z}, w_i, w_j E \text{Div}(A) \]

with multiplication defined as

\[ fg = \sum_{i,j} c_i d_j \text{stand}(w_i) \]

Let J be the ideal in \( \text{Tens}[A] \) generated by

\[ uv = (-1)^{|u|} b^{0}u, \quad u, v E \text{Div}(A) \]

\[ d^{0}d^{0} - (i + j) a^{0}, \quad a E A_{+}. \]

The quotient algebra \( \text{Super}[A] = \text{Tens}[A]/J \) is the superalgebra generated by the signed set \( A \). If \( w E \text{Div}(A) \) is identified with its canonical image in \( \text{Tens}[A] \), then \( \text{mon}(w) \) be the canonical image of \( w E \text{Div}(A) \) in \( \text{Super}[A] \). We often write w in place of mon(w). The superalgebra \( \text{Super}[A] \) is isomorphic to the tensor product \( \text{Super}[A^{+}] \otimes \text{Super}[A^{0}] \otimes \text{Super}[A^{-}] \), where \( \text{Super}[A^{+}], \text{Super}[A^{0}], \) and \( \text{Super}[A^{-}] \) are, respective-
ly, the divided powers algebra, the symmetric algebra, and the exterior algebra generated by $A^+$, $A^0$, and $A^−$.

The module $\text{Super}[A] \otimes \text{Super}[A]$ is given the structure of an algebra (over $\mathbb{Z}$) by setting

$$\left( w \otimes w' \right) \left( u \otimes u' \right) = (-1)^{\text{wt}(w) \cdot \text{wt}(u')} \left( w \otimes u \otimes w' \otimes u' \right),$$

where $w, w', u, u' \in \text{Div}(A)$, and extending by linearity. 

**PROPOSITION 1.** The following rules consistently define a coassociative coproduct $\Delta: \text{Super}[A] \to \text{Super}[A] \otimes \text{Super}[A]$:

(i) if $a \in A^0 \cup A^−$, set $\Delta a = a \otimes 1 + 1 \otimes a$;

(ii) if $a \in A^+$ and $m \in \mathbb{Z}^+$, set $\Delta a^m = a^m \otimes 1 + a^{m-1} \otimes a + \ldots + 1 \otimes a^m$;

(iii) if $a \in A$, let

$$\begin{align*}
(*) \Delta w &= \sum_{i} c_i w^{(1)}_i \otimes w^{(2)}_i \\
\Delta w' &= \sum_{j} d_j w^{(1)}_j \otimes w^{(2)}_j, \\
c_i, d_j & \in \mathbb{Z}; w, w^{(1)}_i, w^{(2)}_i, w', w^{(1)}_j, w^{(2)}_j \in \text{Div}(A),
\end{align*}$$

set

$$\Delta(ww') = \Delta w \Delta w' - \sum_{i,j} c_id_j \text{sign}(w^{(2)}_j) (w^{(1)}_i w^{(1)}_j \otimes w^{(2)}_j w^{(2)}_i),$$

where $\text{sign}(k) = (−1)^k$ for $k \in \mathbb{Z}$.

**PROPOSITION 2.** With the above coproduct, with the augmentation $\epsilon$ defined by $\epsilon(w) = 1$ if $\text{Length}(w) = 0$, $\epsilon(w) = 0$ if $\text{Length}(w) > 0$, and with the antipode $\text{antipode}(w) = \text{sign(Length}(w))w$, the algebra $\text{Super}[A]$ is a Hopf algebra.

**Section 2. Standard Bases**

A signed set $A$ is proper when $A^0 = \emptyset$. If $L$ and $P$ are proper signed sets, the signed set $[L|P]$ consists of all pairs $(x\alpha)$, with $x \in L$, $\alpha \in P$, and with

(i) $(x\alpha) \in [L|P]^+$ if $x \in L^+$ and $\alpha \in P^+$

(ii) $(x\alpha) \in [L|P]^0$ if $x \in L^−$ and $\alpha \in P^−$

(iii) $(x\alpha) \in [L|P]^−$ otherwise.

$\text{Super}[L|P] = \text{Super}[L^+|P^+] \otimes \text{Super}[L^−|P^−] \otimes \text{Super}[L^−|P^+]$, where the factors are, respectively, divided powers, exterior, exterior, and symmetric algebra.

Following Sweedler (10), we write

$$\Delta w = \sum_w w^{(1)} \otimes w^{(2)}$$

as an abbreviation for (*), and more generally

$$\Delta w = \sum_w w^{(1)} \otimes w^{(2)} \otimes \ldots \otimes w^{(n)}$$

for the iterated coproduct ($\Delta_2 = \Delta$). For $w \in \text{Div}(L)$ and $w' \in \text{Div}(P)$, an element $(w|w') \in \text{Super}[L|P]$ is defined by induction over $\text{Length}(w)$ by the following algorithm:

(i) $(w|w') = 0$ if $\text{Length}(w') \neq \text{Length}(w)$

(ii) $(w|w') = (x\alpha)$ if $w = x \in L$ and $w' = \alpha \in P$

The additional properties sometimes used in the definition of the divided powers algebra will not be used in the sequel.

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*The additional properties sometimes used in the definition of the divided powers algebra will not be used in the sequel.*
Let $D$ and $E$ be Young diagrams on $L$ and $P$, such that $a \leq b$ if $a$ and $b$ are successive letters on the same row or column of $D$ and similarly for $E$. One verifies that $\text{Tab}(D|E) \neq 0$ and $\text{Tab}^t(D|E) \neq 0$ (definition below) both hold only if $D$ and $E$ are standard. This remark motivates our choice of the definition of standard Young diagrams.

Section 3. Independence

If $A$ is a proper signed set, the adjoint signed set $A^*$ is a proper signed set together with an isomorphism $\phi: A \rightarrow A^*$ such that $\phi(A^*) = A^*$ and $\phi(A^{**}) = A^{**}$. If $\text{Mon}(\phi): \text{Mon}(A) \rightarrow \text{Mon}(A^*)$ is the induced monoid isomorphism, set $\phi(a) = a^*$ and $\text{Mon}(\phi)(w) = w^*$. For a Young diagram $D = (w_1, ..., w_n)$, set $D^* = \phi(D) = (w_1^*, ..., w_n^*)$.

Set $\text{Tab}^t(D|E) = \text{Tab}(D^*|E^*)$.

**Proposition 5.** For $p \in \text{Super}(L^*[P^*]$ and $q \in \text{Super}(L^*[P^*]$ a bilinear $Z$-valued form is consistently defined by the following algorithm:

(i) $\langle (*a^*a^*), 1 \rangle = (1, (x|x)) = 0$ and $(1, 1) = 1$

(ii) $\langle (*a^*a^*), (x|x) \rangle = \text{sign}(x|x) [a^*a^*]$

(iii) $\langle (*a^*a^*), (y|y) \rangle = 0$ if $y \neq x$ or $a \neq b$

(iv) if $w^* \in \text{Div}(L^*[P^*]$ and $w \in \text{Div}(L[P])$, set $\langle w^*w^*, w \rangle = \sum \text{sign}(w_1) \langle w, w_1 \rangle$.

(v) if $w \in \text{Div}(L^*[P^*]$ and $w^* \in \text{Div}(L[P])$, set $\langle w, w^*w^* \rangle = \sum \text{sign}(w_1) \langle w_1, w^* \rangle$.

**Proposition 6.** Let $w, w' \in \text{Div}(L[P])$, let $u, u' \in \text{Div}(P)$, and let the length of all four words be at least 2. Then,

$$\langle (w^*u^*), (w'|u') \rangle = 0.$$

A matrix $W = \{w_{ij}, i, j = 1, 2, 3, ...\}$ is said to be a Gale–Ryser matrix when

(i) $w_{ij} \in \text{Mon}(A)$ and length($w_{ij}$) $\leq 1$;

(ii) for fixed $i$, almost all $w_{ij} = 1$ and, for fixed $j$, almost all $w_{ij} = 1$.

Under these conditions, the row products:

$$r_i = w_{i1}w_{i2}w_3 ..., \quad i = 1, 2, ...$$

and the column products

$$c_j = w_{1j}w_{2j}w_3 ..., \quad j = 1, 2, ...$$

are well-defined. A Gale–Ryser matrix $W$ interpolates the pair of Young diagrams $D = (w_1, w_2, ...)$ and $D' = (u_1, u_2, ...)$ on $A$ when $\text{cont}(w) = \text{cont}(c)$ and $\text{cont}(u) = \text{cont}(r)$ for all $i$ and $j$. The pair $(D, D')$ is a Gale–Ryser pair when there exists a Gale–Ryser matrix $W$ that interpolates the pair $(D, D')$.

**Proposition 7 (I).** Let $D$ and $D'$ be Young diagrams over $A$ of shape $\lambda$ and $\lambda'$. The pair $(D, D')$ is a Gale–Ryser pair only if $\lambda' \geq \lambda$.

**Proposition 8.** Let $D, D'$ be standard Young diagrams on $L$, and let $E, E'$ be standard Young diagrams on $P$. Then

$$(\text{Tab}^t(D|E), \text{Tab}(D|E)) = \pm 1$$

and

$$(\text{Tab}^t(D'|E'), \text{Tab}(D'|E')) = 0$$

unless both $(D, D')$ and $(E, E')$ are Gale–Ryser pairs.

The uniqueness of the coefficients $c_i$ in Theorem 1 is inferred as follows. From (*)&

$$(\text{Tab}^t(D'|E'), \text{Tab}(D'|E')) = (**) = \sum c_i \langle \text{Tab}^t(D'|E'), \text{Tab}(D|E_i) \rangle.$$
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**THEOREM 2.** When K is of characteristic 0, every joint invariant I of the set T of tensors can be written as

\[ (U, p) = I, \]

where p is a bracket polynomial\(^\text{4} \) in Super[L[P]. When L = L\(^* \), the same conclusion holds over any infinite field K.

Let \( k = \text{Length}(w) \leq n \) for \( w \in \text{Div}(L) \), and suppose that \( \text{cont}(w; a) = 0 \) for \( a \in L_0 \). Let \( a_1, a_2, \ldots, a_{n-k} \in L_0 \cap L_- \); extend the action of the umbral linear functional \( U \) so that each \( a_i \) belongs to a skew-symmetric tensor \( t' \) of step 1. The alternating multilinear form \( \text{cov}(w; t', t', \ldots, t'_{-k}) = (U, [w a_1 a_2 \ldots a_{n-k}]) \) in the variable tensors \( t' \) defines a linear form \( \text{cov}(w) \) on \( \Lambda^{n-k}(V) \).

If \( D = (w_1, w_2, \ldots, w_k) \) is a Young diagram on \( L \), then \( \text{cov}(D) = \text{cov}(w_1) \otimes \text{cov}(w_2) \otimes \ldots \otimes \text{cov}(w_k) \) is a linear form on \( \Lambda^{n-k}(V) \otimes \Lambda^{n-k}(V) \otimes \ldots \otimes \Lambda^{n-k}(V) \). The linear form \( \text{cov}(D) \) is a covariant of the set \( T \).

**THEOREM 3.** If \( \text{char}(K) = 0 \), a linear basis of covariants for the set \( T \) is given by the set \( \text{cov}(D) \), as \( D \) ranges over all standard Young diagrams such that \( \text{cont}(D; a) = 0 \) for all \( a \in L_0 \). If all tensors in \( T \) are skew-symmetric, the same conclusion holds over any infinite field K.

*Application (12, 13)*: Let \( T = \{t\} \), where \( t \) is a skew-symmetric tensor of step 3 and let \( n = 7 \). The following standard Young diagrams give a set of generators for the covariants of \( t \), where all symbols are equivalent:

\[
\begin{align*}
D_0 &= a a a, \\
D_1 &= a a a b b, \\
D_2 &= a a b b c c, \\
D_3 &= a a a b b c, \\
D_4 &= a a b b c c, \\
D_5 &= a a b b d d, \\
D_6 &= a a a b c c, \\
D_7 &= a a b b c c.
\end{align*}
\]

\(D_8 = a a a b b c c \quad b d d e e f \quad c e f f g g g\).

Clearly, \( \text{cov}(D_0) = 0 \) implies \( \text{cov}(D_{i+1}) = 0 \). Let \( c_{i+1} \) be the condition that \( \text{cov}(D_{i+1}) \neq 0 \) but \( \text{cov}(D_i) = 0 \). Each of the conditions \( c_{i+1} \) corresponds biuniquely to one of the canonical forms for the tensor \( t \), to wit, writing

\[
t = a t_1 \wedge t_2 \wedge t_3 + \beta t_4 \wedge t_5 \wedge t_6 + \gamma t_1 \wedge t_4 \wedge t_7 + \delta t_2 \wedge t_5 \wedge t_7 + \epsilon t_3 \wedge t_6 \wedge t_7,
\]

where \( t_i \) are vectors, and where \( \alpha, \ldots, \epsilon \) can take the values 0 or 1, we have

\[
\begin{align*}
c_1 &\text{ iff } \alpha = \ldots = \epsilon = 0, \\
c_2 &\text{ iff } \alpha = 1, \rho = \ldots = \epsilon = 0, \\
c_3 &\text{ iff } \alpha = \gamma = 1, \beta = \delta = \epsilon = 0, \\
c_4 &\text{ iff } \alpha = \delta = \epsilon = 1, \beta = \gamma = 0, \\
c_5 &\text{ iff } \alpha = \beta = 1, \gamma = \delta = \epsilon = 0, \\
c_6 &\text{ iff } \alpha = \beta = 0, \gamma = \delta = \epsilon = 1, \\
c_7 &\text{ iff } \alpha = 0, \beta = \gamma = \delta = \epsilon = 0, \\
c_8 &\text{ iff } \alpha = \beta = \gamma = \delta = 1, \epsilon = 0, \\
c_9 &\text{ iff } \alpha = \beta = \gamma = \delta = \epsilon = 0. \\
c_{10} &\text{ iff } \alpha = \beta = \gamma = \delta = \epsilon = 1.
\end{align*}
\]

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4\(\text{In Weitzenböck's symbolic method (5), the bracket } [a_1 a_2 \ldots a_k] \text{, where } a_i \in L_-, \text{ and } a_i \in L(t') \text{ where } t' \text{ is step 1, is identified with } \langle U, [a_1, a_2, \ldots, a_k]\rangle = \det(t'). \text{ Thus, the same notation is used for both a symmetric and a skew symmetric multilinear form, leading to an inconsistency.}\)

5\(\text{Schouten and Gurevich do not use the symbolic method. The present expressions of Schouten's covariants in symbolic form are new.}\)