Metric rigidity theorems on Hermitian locally symmetric spaces

(semi-negative line bundle/first Chern form/Borel–Weil theorem/Harish–Chandra embedding theorem/compact Kähler manifolds of semipositive curvature)

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Communicated by Hyman Bass, November 4, 1985

ABSTRACT Let \( X = \Omega/\Gamma \) be a compact quotient of an irreducible bounded symmetric domain \( \Omega \) of rank \( \geq 2 \) by a discrete group \( \Gamma \) of automorphisms without fixed points. It is well known that the Kähler–Einstein metric \( g \) on \( X \) carries seminegative curvature (in the sense of Griffiths). I show that any Hermitian metric \( h \) on \( X \) carrying seminegative curvature must be a constant multiple of \( g \). This can be applied to prove rigidity theorems of holomorphic maps from \( X \) into Hermitian manifolds \((V, k)\) carrying seminegative curvature. These results are also generalized to the case of quotients of finite volume. On the other hand, let \((X_\ell, g_\ell)\) be an irreducible compact Hermitian symmetric manifold of rank \( \geq 2 \). Then \( g_\ell \) is Kähler and carries semipositive holomorphic bisectional curvature. I prove that any Kähler \( h \) on \( X \), carrying semipositive holomorphic bisectional curvature must be equal to \( g \), up to a constant multiple and up to a biholomorphic transformation of \( X \).

In the theory of locally symmetric Riemannian spaces \((X, g)\) of negative Ricci curvature, uniqueness theorems for Riemannian metrics \( h \) of seminegative and bounded sectional curvature when \( X \) is of rank \( \geq 2 \) and \((X, g)\) is of finite volume have been proved recently by Eberlein (1), Gromov (unpublished results), and Ballmann and Eberlein (2). In Hermitian geometry there is a more natural notion of negative curvature that applies to Hermitian holomorphic vector bundles. Let \((V, h)\) be a Hermitian holomorphic vector bundle over a complex manifold \( M \) and let \( \Theta = (\Theta_a^b) \) be the curvature form of \((V, h)\), where \( \Theta_a^b = \nabla^2 \Sigma_j \Theta_{a\bar{b}} dz^j dz^\bar{j} \) in local holomorphic coordinates. Then, \((V, h)\) is said to be negative in the sense of Griffiths (3) if \( \Theta_{a\bar{m}} n = \Sigma_a \Theta_{a\bar{b}} n^a n^\bar{b} n^c n^\bar{c} \leq 0 \) whenever \( v = (v^a) \) and \( n = (n^\bar{a}) \) are nonzero. We will say that \((X, h)\) carries negative curvature if and only if the holomorphic tangent bundle \( TX \) is negative in the induced Hermitian metric, also denoted by \( h \). For Kähler manifolds \((X, h)\) this is equivalent to asserting that \((X, h)\) carries negative holomorphic bisectional curvature, a condition weaker than the assertion that \((X, h)\) carries negative Riemannian sectional curvature. The notions of seminegative, positive, and semipositive Hermitian holomorphic vector bundles are similarly defined.

In this article I will state metric rigidity theorems on locally symmetric Hermitian spaces \((X, g)\) of rank \( \geq 2 \) in terms of the notions of seminegativity and semipositivity stated above. Sketches of proofs and applications of such uniqueness theorems and their proofs will be given. Unlike the situation of Riemannian geometry, I have developed a method that applies simultaneously to locally symmetric Hermitian spaces of negative or positive Ricci curvature. In the case of negative Ricci curvature, a major motivation is to study holomorphic mappings from \( X \) into seminagatively curved Hermitian manifolds while, in the case of positive Ricci curvature, I have in mind the classification problem of Kähler manifolds of semipositive holomorphic bisectional curvature.

Theorem 1. Let \((X, g)\) be a locally symmetric Hermitian space of finite volume uniformized by an irreducible bounded symmetric domain of rank \( \geq 2 \). Suppose \( h \) is a Hermitian metric on \( X \) such that \((X, h)\) carries seminegative curvature and \( h \) is dominated by a constant multiple of \( g \). Then \( h = cg \) for some constant \( c > 0 \).

Remark: By the Schwarz lemmas of Royden (4) [in case \((X, h)\) is Kähler] and Chen et al. (5), the condition that \( h \) is dominated by a constant multiple of \( g \) can be obtained by assuming that \((X, h)\) is complete with holomorphic sectional curvature bounded above by a negative constant. Note however that in this hypothesis, \((X, h)\) is not assumed to be complete.

For \((X, g)\) compact, in view of the work of Siu (6, 7), Theorem 1 can be reformulated as follows.

Theorem 1'. Let \((M, h)\) be a compact Kähler manifold of seminegative holomorphic bisectional curvature homotopic to a locally symmetric Hermitian space \((X, g)\) uniformized by an irreducible bounded symmetric domain of rank \( \geq 2 \). Then \((M, h)\) is biholomorphically or conjugate-biholomorphically isometric to \((X, g)\).

As an application of Theorem 1 we obtain the following.

Theorem 2. Let \((X, g)\) be a compact locally symmetric Hermitian space uniformized by an irreducible bounded symmetric domain of rank \( \geq 2 \). Let \((N, h)\) be a Hermitian manifold carrying seminegative curvature. Suppose \( f : X \to N \) is a nonconstant holomorphic mapping. Then, up to a multiplicative constant, 

\[ f \] is an isometry.

In the case of compact Hermitian symmetric spaces \((X_\ell, g_\ell)\) of positive Ricci curvature, we have the following.

Theorem 3. Let \((X_\ell, g_\ell)\) be an irreducible compact Hermitian symmetric space of positive Ricci curvature of rank \( \geq 2 \). Suppose \( h \) is a Kähler metric of semipositive holomorphic bisectional curvature on \( X_\ell \). Then \((X_\ell, h)\) is also Hermitian symmetric.

Remarks: (i) Theorem 3 asserts that \( h = c \Phi^*(g_\ell) \) for some constant \( c > 0 \) and some \( \Phi \in \text{Aut}(X_\ell) \). (ii) The analogue of Theorem 3 for Hermitian metrics \( h \) fails.

Of principal importance to the Hermitian geometry of holomorphic vector bundles \((V, h)\) is the fact that curvatures of Hermitian vector subbundles are smaller than or equal to those of the ambient bundle. From this one deduces the well-known fact that sums of Hermitian metrics of seminegative curvature retain seminegativity, a fact crucial to the proofs of Theorems 1–3.

Proof of Theorem 1: Denote by \( \nabla \) the covariant differentiation on \( X \) defined by the Kähler–Einstein metric \( g \). We can regard the metric \( h \) as a covariant tensor of type \((1, 1)\). By the irreducibility of \((X, g)\), to prove Theorem 1 it suffices to show that \( \nabla h = 0 \) on \( X \). The Hermitian holomorphic vector bundle \((TX, g)\) corresponds to a Hermitian holomorphic line bundle \((L, \xi)\) on \( PT(X) \) such that the first Chern form \( c_1(L, \xi) \) is negative semidefinite (cf. ref. 3). Furthermore, \( c_1(L, \xi) \) has a zero eigenvalue at \((x, [\eta]) \in PT_x \) if and only

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if $R^{\alpha\beta}_{\gamma\delta} = 0$ for some $\zeta \neq 0$ for the curvature tensor $R^\alpha_{\beta\gamma\delta}$ of $g$. Since $X$ is of rank $\geq 2$ at each $x \in X$ there exists such an $\eta \in T_x$. Suppose $\mathcal{F}$ is a $k$-dimensional complex submanifold of $PT(X)$ such that at every point of $\mathcal{F}$ there exist exactly $s$ zero eigenvalues of $c_1(L_g, g)$ and let $\nu_m$ be a semipositive $(m, m)$ form on $\mathcal{F}$. Then,

$$
\int_{\mathcal{F}} [-c_1(L_g, g)^{k+1} + \Lambda \nu_m] = 0.
$$

[1]

Since the integral is a topological invariant of the line bundle $L_g$, we have

$$
\int_{\mathcal{F}} [-c_1(L_g, g + \hat{h})^{k+1} + \Lambda \nu_m] = 0.
$$

[2]

Since the integrand is pointwise nonnegative it must vanish identically on $\mathcal{F}$. For a suitable choice of $\nu_m$ and $\Lambda$ we can show then that $c_1(L_g, g + \hat{h})$ has exactly $s$ zero eigenvalues everywhere. This forces the vanishing of many bisectional curvature $R^\alpha\beta_{\gamma\delta\eta\xi}$ of $(X, g + \hat{h})$ for which $R^\alpha\beta_{\gamma\delta\eta\xi} = 0$. Comparing the two expressions we can conclude that $\nu_m(A) = 0$ for any $A \in T(X)$ at the point. With a sufficiently large set of $(\sigma, \mu)$ we can show that $\nu_m = 0$ on $X$, proving Theorem 1.

To construct $\mathcal{F}$ let $\mathcal{M} \subset PT(X)$ be defined by unit vectors $A \in T_X$ such that $|R^\alpha\beta_{\gamma\delta\eta\xi} = 0$ and define $\mathcal{F} = \mathcal{M} = \cup_{A \in \mathcal{M}}$. For $X$ of rank $\geq 2$ and locally integrable the Hermitian form $H_{\xi, \eta}(\xi, \eta) = R^\alpha\beta_{\gamma\delta\eta\xi}$ has exactly $s$ $p(X, g)$ - 1 zero eigenvalues, where $p(X, g)$ is the degree of (strong) non-degeneracy of bisectional curvatures of the covering domain $X$ computed by Siu (7) and by Zhong (8). Write $X_0 = G_0/K$ where $G_0$ is semisimple and denote by $\sigma$ the coset $eK$. We can prove that $\mathcal{M} \subset \mathcal{M}_g$ and if only if $\sigma$ is a dominant real vector of the irreducible $K$-representation space $T_0$, so that $\mathcal{M}$ is a homogeneous complex submanifold of $PT(X)$ by the Borel-Weil theorem (10). Let $X_0 \subset X$ be the Borel embedding of $X_0$ into its compact dual $X = G/K$. Then the complexified Lie group $G^C$ acts on $PT(X)$. Using the complex analyticity of $\mathcal{M}$ and the Borel embedding theorem, we can show that $\mathcal{M} = \cup_{A \in \mathcal{M}}$ is precisely $G^C$($\{a\}$) for any $a \in \mathcal{M}$, so that $\mathcal{M} = \mathcal{M}(X$) is a complex manifold. Furthermore, we can check by using the Harish-Chandra embedding $X_0 \subset C^C$ that the zero eigenvectors of $c_1(L, g)$ at $[a] \in \mathcal{M}$ are tangent to $\mathcal{F} = \mathcal{M}(X)$. Choosing $\alpha(x)$ to be the $(n - 1)$th exterior power of the pull-back of the Kähler form of $(X, g)$, we can show that $\nu_m = 0$ for all $[a] \in \mathcal{M}$ and all $\xi \in T_x$ such that $R^\alpha\beta_{\gamma\delta\eta\xi} = 0$, $\xi \in \mathcal{T}_x$. Finally, by expanding $\alpha$ and $\tau$ we can verify the identity $\nu_m = 0$, thus proving Theorem 1.

Proof of Theorem 2: One simply applies Theorem 1 to $(X, g + f^*(h))$.

Proof of Theorem 3: For $(X_0, g_0)$ a compact Hermitian symmetric space, the cotangent bundle $(T^*(X_0), g^*)$ is a Hermitian vector bundle of semisemigative curvature. Let $(A, g^*)$ be the corresponding Hermitian line bundle on $T^*(X_0)$. Then $c_1(A, g^*)$ is negative semidefinite everywhere. Let $\mathcal{M}(X_0)$ be defined similar to $\mathcal{M}(X)$ in Theorem 1. In terms of the Borel embedding $X_0 \subset X$, one can choose $g_0$ so that $(X_0, g_0)$ is an isometry at the origin $0$ and the curvature tensor of $(X_0, g_0)$ at $0$ is simply opposite to that of $(X, g)$.

One can then prove that $\mathcal{M}(X_0) \subset \mathcal{M}(X) \subset PT(X)$. For every $\tau \in T_x$, we denote by $T^\tau \subset T_x$ the lifting of $\tau$ by the contravariant metric tensor $(\cdot, \cdot)$. Let $\mathcal{T} = \mathcal{T}^*(X) \subset PT(X)$ be thus defined by lifting. Applying the argument of Theorem 1 to $\Delta$ and $\mathcal{F} = \mathcal{M}(X_0)$, we can prove that $\nu_m = 0$ for all $\tau \in M^*, R^\alpha\beta_{\gamma\delta\eta\xi} = 0$ for the curvature tensor $R^\alpha\beta_{\gamma\delta\eta\xi}$ of $(X_0, g_0)$, $\tau \in T_x$ arbitrary. The $c$-linear subspace $W \subset T_\tau \subset T_x$ spanned by $g \otimes \tau$ is the proper subspace of zero eigenvectors of the quadratic form $P$ on $T_{x} \otimes T_x$ defined by $P(\eta \otimes \tau)$.
niques of Jost and Yau (14) and Mok (15), we can also obtain a nontrivial holomorphic map \( f : X \to S \) into some compact Riemann surface of genus \( \geq 0 \). By Theorem 2 (and an easy generalization) we obtain a contradiction unless real rank \( f \) = 1. But then \( f \) would map \( X \) onto a closed geodesic of \( Y \), yielding a nontrivial map of \( \pi(X)/[X, X] = H(X) \) into \( Z \). This contradicts the well-known fact that the first Betti number \( X \) vanishes.

**Proof of Theorem 5:** By the results of Borel (16), there exists a compact quotient \( X = \Omega / \Gamma \) of \( \Omega \). The Hermitian metric \( h \) on \( \Omega \) induces an upper-semicontinuous Finsler metric \( s \) on \( X \) by the covering map \( \pi : \Omega \to X \) by taking suprema over preimages. \( s \) can be smoothed to yield an \( X \) smooth Finsler metric \( \sigma \) of negative curvature. Recall that \( L \) denotes the holomorphic line bundle on \( PT(X) \) induced by the holomorphic tangent bundle \( T(X) \to X \). \( s \) corresponds to a Hermitian metric of negative curvature on \( L \to PT(X) \). The integral formula (Eq. 2) on \( \mathcal{M}(X) \) with \( s \) replacing \( \sigma \) given in the proof of Theorem 1 yields immediately a contradiction unless \( \Omega \) is of rank 1.

Regarding the geometry of compact Kähler manifolds of semipositive holomorphic bisectional curvature, there is the following general conjecture.

**Conjecture.** Let \((X, h)\) be a compact Kähler manifold of semipositive holomorphic bisectional curvature and positive Ricci curvature. Then \((X, h)\) is biholomorphically isomorphic to \((X_1, h_1)x-x\mathbb{R}^m, h_2\), where \(X_i, 1 \leq i \leq m\), is the underlying complex manifold of some irreducible Hermitian symmetric space and \(h_i\) is a Hermitian symmetric space on \(X_i\) unless \(X_i\) is a projective space. Based on the statement and the method of proof of Theorem 3, we obtain the following.

**Theorem 6.** Let \(X = X_1x-x\mathbb{R}^m\) be a compact Hermitian symmetric space and \(h\) be a Kähler metric on \(X\) carrying semipositive holomorphic bisectional curvature. Then \((X, h)\) splits isometrically into \((X_1, h_1)x-x\mathbb{R}^m, h_2\).

**Theorem 7.** The conjecture above is true for \(X\) of complex dimension \(n \leq 3\).

**Proof of Theorem 6:** It suffices to take \(X = X' \times X''\). Let \((e_i)_{i \in \mathbb{N}}\) be local holomorphic coordinates at \(x' \in X', \ x'' \in X''\). By an integral formula similar to but much more elementary than the one used in the proof of Theorem 3, we can deduce that \(\partial h/\partial z_\beta = \partial h/\partial z_\alpha = 0\) at \((x', x'')\) for \(1 \leq i \leq l, 1 \leq j \leq n\). Theorem 6 asserts that \(\partial h/\partial z_\beta = \partial h/\partial z_\alpha = 0\) at \((x', x'')\). This is obtained from the preceding equations, the Kähler condition \(\partial h/\partial z_\alpha = \partial h/\partial z_\beta\), and the equation \(\partial (\partial h/\partial z_\alpha) (\partial h/\partial z_\beta) = \partial (\partial h/\partial z_\alpha) \partial h/\partial z_\alpha = 0\).

**Proof of Theorem 7:** Using the results of Howard and Smyth (17), Mori (18), Siu and Yau (19), Siu (20), and Bando (21), Bando has proven that, for \(n \leq 3\), \(X\) is biholomorphic to a compact Hermitian symmetric space. The conjecture for \(n \leq 3\) then follows from Theorems 3 and 6.

Finally, I record a theorem on the geometry of projective submanifolds that is motivated by study of the curvature tensor of Hermitian symmetric spaces. Let \(X\) be an \(n\)-dimensional compact Hermitian symmetric space. Recall that \(\mathcal{M} \subset PT(X)\) is the set of all \([x] \in PT(X)\) such that \(R_{x \alpha \beta \gamma} \) realizes the maximum holomorphic sectional curvature at \(x\). Ig-