Reduction, the trace formula, and semiclassical asymptotics

(elliptic operators/spectral theory/Fourier integral operators)

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ABSTRACT We state a theorem that relates the theory of dimensional reduction in Hamiltonian mechanics to the spectral properties of elliptic operators with symmetries on compact manifolds. As an application, we show that the spectrum of the Schrödinger operator, \(-\Delta + V\), as \(\hbar \to 0\), contains geometric information about the closed trajectories of a classical particle with Hamiltonian \(\|p\|^2 + V(q)\). More generally, we show that this is true for particles with internal degrees of freedom and subject to an external Yang–Mills field, the classical limit being the Wong–Sternberg–Weinstein system for such particles.

As an introduction, we begin by stating a special case of Theorem 3. Let \(\lambda_j(h) \leq \lambda_{j+1}(h) \leq \lambda_{j+2}(h) \leq \ldots\) be the eigenvalues (with multiplicities) of the Schrödinger operator \(-\hbar^2\Delta + V\) on a compact Riemannian manifold \(M\), where \(V\) is smooth and positive. Let \(E > 0\), and consider the distribution of the eigenvalues \(\lambda_j(h)\) that are approximately equal to \(E^2\), as \(\hbar \to 0\). By the correspondence principle, one expects that such eigenvalues are closely related to the movement of a classical particle under the potential \(V\) and with energy \(E^2\). One way to make this precise is as follows. Let \(\varphi\) be a fixed non-negative Schwartz function on the real line satisfying \(\int \varphi(x)dx = 1\), let

\[\mu_{m,j} = m(\lambda_j(1/m))^{1/2}, \quad m, j \in \mathbb{Z}^+\],

and consider the weighted spectral measure on \(\mathbb{R}\),

\[\sigma_E = \sum_{m,j=1}^\infty \varphi(\frac{\mu_{m,j}}{E} - m) \delta\left(\mu - \frac{\mu_{m,j}}{E}\right),\]

where, by virtue of the rapid decrease of \(\varphi\), those eigenvalues such that \(\lambda_j(1/m)\) is close to \(E^2\) contribute much more heavily than the rest. We are interested in the (inverse) Fourier transform of \(\sigma_E\),

\[Tr = \sum_{m,j=1}^\infty \varphi(\frac{\mu_{m,j}}{E} - m) \exp\left(it \frac{\mu_{m,j}}{E}\right)\].

**Theorem 1.** Assume that \(\varphi\) is the Fourier transform of a function with compact support. Then \(Tr\) is a tempered distribution, and its singularities are contained in the set of periods of the closed trajectories of the Hamiltonian system on \(T^*M - \{0\}\) with Hamiltonian function \(H = (b + V)^{1/2}\), where \(b\) is the square of the Riemannian norm function, on the energy surface \(\{H = E\}\). Moreover, if such periodic trajectories are nondegenerate, and if \(\gamma_1, \ldots, \gamma_k\) are all the trajectories with a given period \(T\), then we can write

\[Tr = C_T \delta(t - T) + f_T,\]

where \(f_T\) is locally in \(L^1\) in a neighborhood of \(T\), and

\[C_T = \sum_{j=1}^k \frac{T(j)\# \exp\left[i\sigma(j)/4\right]}{2\pi |l - p_j|^{1/2}},\]

where \(T(j)\#\) is the primitive period of \(\gamma_j\), \(p_j\) is its linearized Poincaré map, and \(\sigma(j)\) is its Maslov index.

For an earlier, nonrigorous version of a theorem of this kind, see ref. 1. More recently, Chazarain (2) proved a result analogous to the first half of Theorem 1. Just as in ref. 3, the assumption that the closed trajectories on \(\{H = E\}\) are nondegenerate can be generalized to include the case when they form "clean" submanifolds. \(Tr\) always has a classical conormal singularity at \(t = 0\), the study of which leads to Weyl-type estimates. Notice that if we work with a slight modification of \(Tr\), namely \(\hat{Tr}(t) = Tr(2Et)\), then we obtain the more pleasing result that the singularities of \(\hat{Tr}\) are related to the closed trajectories of the classical Hamiltonian \(H^2 = b + V\) on the energy surface \(\{H^2 = E^2\}\). However, our version of Theorem 1 is more natural from the point of view of Theorem 2.

We will generalize this theorem to particles with internal degrees of freedom and subject to a Yang–Mills field (see Theorem 3 below), replacing \((T^*M, H^2)\) by the Wong–Sternberg–Weinstein Hamiltonian system for such particles. Theorems 1 and 3 follow from a general trace formula for elliptic operators with symmetries, which relates the quantum and the classical theories of reduction. We turn to this result next.

**Reduction.** Let \(N\) be a compact Riemannian manifold, and \(P\) an elliptic, self-adjoint, first-order pseudodifferential operator on \(N\) with positive symbol. As is well-known, the spectrum of \(P\) consists entirely of eigenvalues accumulating at \(+\infty\) and nowhere else. In many interesting cases the operator \(P\) admits symmetries, in the following sense: there is a Lie group \(G\) and a representation, \(\rho\), of \(G\) in \(L^2(N)\) such that \(P\) commutes with every operator \(\rho(g), g \in G\). The representation \(\rho\) breaks \(L^2(N)\) into an orthogonal Hilbert space sum,

\[L^2(N) = \bigoplus_{\gamma \in \Gamma} W_{\gamma},\]

where, for every isomorphism class of irreducible representations, \(\gamma, W_{\gamma}\) is the space of functions on \(N\) that transform, under \(\rho\), according to \(\gamma\). Since \(P\) commutes with \(\rho\), \(P\) preserves the previous decomposition of \(L^2(N)\), and hence we can consider the eigenvalues (listed with multiplicities) \(\mu_{\gamma,1} \leq \mu_{\gamma,2} \leq \mu_{\gamma,3} \leq \ldots\) of the restriction of \(P\) to \(W_{\gamma}\). (At this stage this list might be finite or empty for a given \(\gamma\).

The representation \(\rho\) is often a "quantization" of symmetries of the classical dynamical system corresponding to the one-parameter unitary group generated by \(P\). Explicitly, let \(X\) denote the symplectic manifold \(T^*N - \{0\}\). The principal symbol, \(\rho\), of \(P\) is a real-valued, positive-homogeneous smooth function on \(X\) that generates a Hamiltonian flow, \(\{\phi_t\}\). The classical Hamiltonian system \((X, \{\phi_t\})\) admits the
group $G$ as a symmetry group if there is a Hamiltonian group action of $G$ on $X$ that commutes with $(\phi_t)$. We shall assume that this is the case; we will denote the action of $g \in G$ on $x \in X$ by a dot $(g \cdot x)$, and the moment map of the action, which we assume to be positive-homogeneous, will be denoted by

$$\Phi : X \to g^*,$$

where $g$ is the Lie algebra of $G$ and $g^*$ is its dual. We will assume that the representation $\rho$ is a quantization of the given Hamiltonian action, in the following sense. We can think of the representation $\rho$ as a single operator, $\rho : C^\infty(N) \to \mathcal{C}^\infty(G \times N)$. Our assumption is that $\rho$ is a Fourier integral operator associated to the following canonical relation (the moment Lagrangian):

$$\Gamma = \{(g, \gamma; x; g^* \cdot x) | \gamma = \Phi(x) \} \subset T^*G \times X \times X.$$

Here we have made the identification $T^*G \cong G \times g^*$, using the right translations on $G$. One case in which this assumption is automatically satisfied is when $p$ arises from an action of $G$ on the manifold $N$, for then the Schwartz kernel of the operator $\rho$ is just the Dirac delta function along the graph of the action (in $G \times N \times N$), and $\Gamma$ is the conormal bundle of this graph.

From the classical point of view, the operation that corresponds to restricting $\rho$ to $W$, ought to be the Marsden–Weinstein reduction of $X$. Let $G_u$ be the coadjoint orbit of $G$ associated to $u$ by the Böll–Borel–Weyl–Kostant theorem (see ref. 4). Our goal is to understand this correspondence. Let $O \subset g^*$ be an integral coadjoint orbit of $G$. Since $\Phi$ is equivariant, the group $G$ acts on $\Phi^{-1}(O)$. As a manifold, the Marsden–Weinstein reduction of $X$ with respect to $O$ is the quotient space

$$X_0 = \Phi^{-1}(O)/G.$$

If $\Phi$ intersects $O$ transversely, this space is a $V$-manifold in the sense of Satake; for simplicity we will make the additional assumption that $X_0$ is in fact an ordinary manifold. Since $p$ is $G$-invariant, its restriction to $\Phi^{-1}(O)$ descends to a smooth function, $\rho \in \mathcal{C}^\omega(X_0)$. The manifold $X_0$ has various other descriptions (see below), which show that $X_0$ has a symplectic structure in a natural way. Then the Marsden–Weinstein reduction of $(X, \{\phi_t\})$ is $(X_0, \{\phi_t\})$, where $\{\phi_t\}$ is the Hamiltonian flow of $\rho$ in $X_0$, and the natural projection $\Phi^{-1}(O) \to X_0$ intertwines $\{\phi_t\}$ and $\{\phi\}$. We will shortly state a trace formula involving the eigenvalues $\mu_{\alpha,j}$. It is the closed trajectories of $\{\phi_t\}$ that contribute to our trace formula; however, to describe how they contribute to it, we need a different description of $X_0$. We will see that there is a canonical circle bundle with connection over $X_0$, and that is the logarithm of the holonomy of a closed trajectory of $\{\phi_t\}$ that gives rise to a singularity of our trace. From a physical point of view, the logarithm of the holonomy of a path in $X_0$ is to be interpreted as the action of the path; so the singularities of our trace are at the values of the action of the closed trajectories of $\{\phi_t\}$.

Let $\alpha \in O$, and let $G_\alpha$ denote the isotropy group of $\alpha$ with respect to the coadjoint representation. Again by equivariance of $\Phi$, $G_\alpha$ acts on $\Phi^{-1}(\alpha)$, so we can form the quotient space $X_\alpha = \Phi^{-1}(\alpha)/G_\alpha$. It is easy to see that for $\alpha \in O$ there is a natural identification of $X_\alpha$ with $X_0$. Hence our previous assumption of $X_0$ being an ordinary manifold will be satisfied if we assume that the action of $G_\alpha$ on $\Phi^{-1}(\alpha)$ is free, which we now do. The fact that the orbit $O$ is integral means that there is a unitary character, $\chi_\alpha : G_\alpha \to S^1$, such that for every $\xi \in g_\alpha$ (the Lie algebra of $G_\alpha$) one has

$$d\chi_\alpha(\xi) = 2\pi i \langle \alpha, \xi \rangle.$$

where $\langle , \rangle$ denotes the dual pairing of $g$ and $g^*$. By the associated bundle construction applied to the $G_\alpha$-bundle $\Phi^{-1}(\alpha) \to X_\alpha$, we see that there is a naturally defined principal $S^1$-bundle over $X_\alpha$. Using the identification of $X_\alpha$ with $X_0$, we obtain a principal circle bundle over the Marsden–Weinstein space, $\pi : X \to X_0$. Next we show that there is a natural connection on $\pi$, whose curvature form is the symplectic form on $X_0$; that is, $\pi$ is a prequantization of $X_0$.

Let $\eta$ denote the canonical one-form on $X$ (i.e., $\Sigma_\rho dq_\rho$) restricted to $\Phi^{-1}(\alpha)$. Since the moment map is homogeneous, it is easy to see that the form $\eta$ is $G_\alpha$-invariant and furthermore that $\forall \xi \in g_\alpha$, if $\xi \not\in \{\phi_t\}$ generated by $\xi$, then

$$\eta(\xi) = \langle \alpha, \xi \rangle.$$

In other words, by refs. 3 and 4 we have

$$2\pi i \eta(\xi) = d\chi_\alpha(\xi).$$

This easily implies that the form $\theta = 2\pi i \eta + d\theta$, on $\Phi^{-1}(\alpha) \times S^1$, descends to a connection form, which we also denote by $\theta$, to the associated bundle

$$Z = \{\Phi^{-1}(\alpha) \times S^1\}/G_\alpha,$$

Since $d\theta = 2\pi d\eta$, it is clear that the curvature of $\theta$ is $2\pi \omega$, where $\omega$ is the symplectic form of $X_0$.

A closed trajectory, $\gamma$, of $\{\phi_t\}$, has three numbers associated to it; we will now describe these and introduce some notation for them. The first is its period, $T(\gamma) \in \mathbb{R}$. The second is its holonomy, $\omega(\gamma) \in S^1$, with respect to the connection on $\pi : X \to X_0$ we just described. The third, $\Theta(\gamma) \in S^1$, is what we will call the Hamiltonian holonomy of $\gamma$ and is defined as follows. Let $y \in Y$—that is, $y \in X_0$ is such that $\Phi(y) = y$. Pick $\alpha \in O$ and let $x = \Phi^{-1}(\alpha)$ be above $y$. Then there is a unique $\gamma \in G_\alpha$ such that $\gamma \cdot \Phi(y) = x$. By definition, the Hamiltonian holonomy of $\gamma$ is $\Theta(\gamma) = \chi_\alpha(g)$. (It is easy to see that it only depends on the path $\gamma$, and not on the choices of $\gamma$, $\alpha$, and $x$.) These three numbers are not unrelated; in fact one has the following formula: if $\gamma$ is a closed trajectory of $\{\phi_t\}$ in the energy surface $\{\beta = E\}$, then $\omega(\gamma) = \Theta(\gamma)$ exp$(iT(\gamma))$.

We can now state Theorem 2. For every positive integer $m$, let $mr$ be the irreducible representation of $G$ corresponding to the integral coadjoint orbit $mO$, and fix a positive number $E$; we call the sequence $(mr)$ a ladder of representations (see ref. 5). Consider the eigenvalues, $\mu_{m,j}$, that are approximately equal to $mE$, for all $m$ and $j$. Specifically, let $\varphi$ be a non-negative Schwartz function on the real line, with $\int \varphi(t)dt = 1$, and consider the distributional function of $t \in R$,

$$\text{Tr} = \sum_{m,j} \varphi \left( \mu_{m,j} - m \right) \exp \left( i \frac{\mu_{m,j}}{E} \right).$$

**Theorem 2.** Assume that $\varphi$ is the Fourier transform of a function $\psi$ with compact support contained in $(-\pi, \pi)$, and let $Y$ denote the set of closed trajectories of the reduced flow $\{\phi_t\}$ on the energy surface $\{\beta = E\} \subset X_0$. Then the singular support of the distribution $\text{Tr}$ is contained in the set of real numbers $1/2 \log |\omega(y)|$, where $\gamma \in Y$. Moreover, if the trajectories in $Y$ are nondegenerate and a given singularity, $\alpha$, corresponds to finitely many closed trajectories, $\gamma_1, \ldots, \gamma_k$, then one can write $\text{Tr}$ as a sum

$$\text{Tr} = \text{Tr}_a + \text{Tr}_a,$$
where \( f_a \) is in \( L^1_{\text{loc}} \) in a neighborhood of \( a \) and \( e_a = C_a \delta(t - a) \), with

\[
C_a = \frac{1}{2\pi} \psi(\theta) \left( \frac{\exp(\pi \sigma(j)/4)}{|1 - P_j|^{1/2}} \right).
\]

Here \( T(j) \) is the primitive period of \( \gamma, P_j \) is its linearized Poincaré map, \( \sigma(j) \) is a suitable Maslov index attached to \( \gamma_j \), and \( i\theta_j \) is the logarithm of \( O(\gamma_j) \).

The distribution \( T \) always has a classical conormal singularity at \( t = 0 \), the study of which leads to asymptotic estimates on the counting function

\[
N(\mu) = \sum_{m=1}^{\infty} \sum_{\mu_m \leq \mu} \varphi \left( \frac{\mu_m}{E} - m \right).
\]

As in ref. 3, the assumption that the orbits in \( Y \) be nondegenerate can be generalized; the closed trajectories can form "clean" submanifolds of \( X_0 \).

A Semiclassical Trace Formula. Let \( \pi: M \rightarrow B \) be a principal \( K \)-bundle, where both \( K \) and \( M \) are compact. Suppose we are given a positive-definite Riemannian metric on \( B \) and a connection on \( \pi \), which we think of as a generalized Yang–Mills field on \( B \). Moreover, on \( M \) we consider a \( K \)-invariant, smooth, real-valued function \( V \). If \( \tau \) is a unitary representation of \( K \) on a finite-dimensional vector space \( V \), then the above data define a Schrödinger operator,

\[
S_{\tau} = -h^2 \Delta_{\tau} + V,
\]

on sections of the associated vector bundle \( M \times_{\tau} V \). The operator \( S_{\tau} \) has a self-adjoint extension to the Hilbert space of \( L^2 \) sections of \( M \times_{\tau} V \), and its spectrum consists entirely of eigenvalues clustering at \( +\infty \) and nowhere else. We list these eigenvalues in increasing order, with multiplicities:

\[
\lambda_{\tau,1}(\hbar) \leq \lambda_{\tau,2}(\hbar) \leq \ldots.
\]

These are the energy levels of a quantum particle whose universe is \( B \) and which is subject to the given Yang–Mills and scalar potentials.

A representation \( \tau \) of \( K \) generates a ladder of representations, \( (m\tau) \). For simplicity of notation, we will write \( \lambda_{m,j}(\hbar) \) for the \( j \)th eigenvalue of \( S_\tau \) with \( \tau = m\tau \). Let \( \varphi \) and \( E \) be as before. Choose a bi-invariant metric on \( K \). The corresponding Laplacian, \( \Delta_K \), on \( K \), acts (as an element in the center of the enveloping algebra of \( K \)) by multiplication by a constant, \( c_m \), in the representation space of \( \tau_m \). Now consider

\[
Tr = \sum_{m,j=1} \varphi \left( \frac{\mu_m}{E} - m \right) \exp \left( it \frac{\mu_m}{E} \right),
\]

where

\[
\mu_{m,j} = [m^2 \lambda_{m,j}(1/m) + c_m]^{1/2}.
\]

The trace formula for \( Tr \) is in terms of the closed trajectories of the Hamiltonian system describing the movement of a classical particle with internal symmetry group \( K \) and subject to the Yang–Mills potential given by the connection on \( \pi \), and to the scalar potential \( V \). We will call this system the WSW system, after Wong (6), Sternberg (7), and Weinstein (8); we now recall its definition. Let \( O \) be the coadjoint orbit of \( K \) corresponding to the representation \( \tau \), and let \( X_O \) be the Marsden–Weinstein reduction of \( X = T^*M \) with respect to \( O \). The connection on \( \pi \) defines a fibration

\[
F:X_O \rightarrow T^*B
\]

with fiber \( O \). Then the WSW system in question is the Hamiltonian system with phase space \( X_O \) and Hamiltonian, \( H^2 \), equal to the pull-back of \( F \) of \( b + V \), where \( b: T^*B \rightarrow R \) is the square of the Riemannian norm on \( B \). We can now state our semiclassical trace formula.

**Theorem 3.** Let \( ||O|| \) be the norm of (any element of) \( O \) with respect to the chosen metric on \( K \). Then the singularities of \( Tr \) are related to the closed trajectories of \( \hat{p} = (H^2 + ||O||^2)^{1/2} \) on the energy surface \( Y_E = \{H^2 = E^2 - ||O||^2\} \subset X_0 \), in the sense of Theorem 2.

Notice that, on \( Y_E \), \( dH^2 = 2E \ d\hat{p} \), and so a closed trajectory of \( \hat{p} \) of period \( T \) on \( Y_E \) is (up to a constant parametrisation) a closed trajectory of \( H^2 \), of period \( T/2E \). Theorem 3 follows from Theorem 2 by taking \( N = M \times S^1 \), \( G = K \times S^1 \), and \( \Delta = (\Delta - \nabla^2 g)^{1/2} \), where \( \Delta \) is the Laplacian on \( M \) corresponding to the Kaluza–Klein metric defined by the connection on \( \pi \) and the metrics on \( B \) and \( K \) (see ref. 9, section 5, for additional information). Obviously Theorem 1 is the special case of Theorem 3 when \( K \) is the trivial group.

The type of Weyl asymptotics that one obtains by studying the singularity of \( Tr \) at \( t = 0 \) has been considered in the literature. We refer to refs. 10–12 and a forthcoming paper by R. Schrader and M. Taylor.

**Concluding Remarks.** Theorem 2 implies some of our previous results (9, 13). It has several other applications; for example, to the study of completely integrable systems and to the theory of band asymptotics. These applications, as well as the details of the proof of Theorem 2 (in particular the exact meaning of the Maslov factors), will appear elsewhere.

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