Correction. In the article "Direct observation of large chiral domains in chloroplast thylakoid membranes by differential polarization microscopy" by Laura Finzi, Carlos Bustamante, Gyöző Garab, and Ching-Bo Juang, which appeared in number 22, November 1989, of Proc. Natl. Acad. Sci. USA (86, pp. 8748–8752), Fig. 2 was incompletely reproduced due to a printer's error; the more accurate Fig. 2 appears below.

Fig. 2. (a) Nonpolarized image of two edge-aligned chloroplasts; (b) LD image $I_H = I_V/I_H + I_V$: horizontal polarization (H) is defined parallel to the bottom edge of this figure. The largest positive value of the dichroic ratio measured in this LD image was 0.11. Color scale for the dichroic values is shown. For this image the lightest shade of blue in the color bar corresponds to a dichroic ratio of 0.11. (Bar = 2 μm.)

Correction. The article "Texture interactions determine perceived contrast" by Charles Chubb, George Sperling, and Joshua A. Solomon, which appeared in number 23, December 1989, of Proc. Natl. Acad. Sci. USA (86, 9631–9635), was listed under the wrong classification in the Table of Contents. The article should have been listed under Psychology instead of Population Biology.

Correction. For the article "A case at last for age-phased reduction in equity" by Paul A. Samuelson, which appeared in number 22, November 1989, in Proc. Natl. Acad. Sci. USA (86, 9048–9051), the following printer's error should be noted. The incorrect listing in the Table of Contents was Biological Sciences. The correct category is Social Sciences.
A case at last for age-phased reduction in equity
(random walk/expected utility/subsistence consumption–wealth)

PAUL A. SAMUELSON
Department of Economics, Massachusetts Institute of Technology, Cambridge, MA 02139
Contribution by Paul A. Samuelson, June 7, 1989

ABSTRACT Maximizing expected utility over a lifetime leads one who has constant relative risk aversion and faces random-walk securities returns to be "myopic" and hold the same fraction of portfolio in equities early and late in life—a defiance of folk wisdom and casual introspection. By assuming one needs to assure at retirement a minimum ("subistence") level of wealth, the present analysis deduces a pattern of greater risk-taking when young than when old. When a subsistence minimum is needed at every period of life, the rentier paradoxically is least risk tolerant in youth—the Robert C. Merton paradox that traces to the decline with age of the present discounted value of the subsistence-consumption requirements. Conversely, the decline with age of capitalized human capital reverses the Merton effect.

DEBATED AGE EFFECTS

Folk wisdom recommends that we should be more risk tolerant when young, reducing as we approach retirement the fraction of wealth put into risky equities and increasing our safe-cash exposure. However, Samuelson (1) found that a rational maximizer of expected utility, even though subject to constant relative risk aversion and facing random-walk securities returns, would rationally invest the same fraction in equities at all ages. This result held despite the young person's "‘having more opportunities and time to recoup from initial bad luck;'" and despite the law of large numbers guaranteeing a tendency for securities to live up over many repeated independent periods to their intrinsic superior profitability over safe-cash assets. Leland (2), Mossin (3), Merton (4), and Hakansson (5) arrived at this same null result.

By dropping the assumption of white-noise martingales in favor of mean-reverting negative serial correlation, Samuelson (6, 7) was able to confirm, however, the qualitative wisdom of folklore for investors with constant relative risk aversion exceeding that of Bernoulli's logarithmic utility. (For $U = -1/W^y$, the theorem obtains that under mean reversion, many-period investors will be more risk-taking than few-period investors.)

Also, Samuelson (ref. 1, p. 245) had noted the uncontroersial result concerning "‘businessman's risk.’" Suppose I face annual earnings from work, known to be constant over preretirement years. The capitalized (present discounted) value of this human capital declines as I age and as my time to retirement shrinks. Therefore, if I hold equities in constant fraction to my total wealth—defined as portfolio wealth plus human capital—my observed portfolio fraction in equities will be seen to decline rationally with age. (With negative human capital, as when I owe a specified number of periodic cash installments, the reverse of the folk-wisdom pattern is of course entailed.)

The present analysis deduces a third, new, case for age-phased riskiness reduction. It retains the random-walk asumption of uniform and independent probability distributions, and it continues to ignore human-capital complications. Its new element is recognition of the realistic fact that many people save toward assuring for themselves or their heirs a certain minimum ("‘subistence’) level of wealth. It is proved here that, when we replace $U = \log W$ or $U = W^y/\gamma$ by

$$U = \log[W - S] \text{ or } (W - S)^y/\gamma, \quad 0 \neq \gamma < 1, \ S > 0, \ [1]$$

where $U = \text{utility}$, $W = \text{terminal wealth}$, $S = \text{minimum terminal wealth insisted on}$, and $\gamma = \text{parameter of relative risk tolerance}$, then the fraction rationally allocated to equities decreases with age toward retirement, confirming folk wisdom.

Zvi Bodie and William Samuelson have deduced (7) an independent new reason for greater risk tolerance when young. It stems from the fact that I can plan to work harder if my risks early turn out adversely.

It will suffice for the present demonstration to consider the simplest case of a single risky stock and safe cash. However, the result and its proof are general, and I provide for the reader the illustrative exact formulas that hold for the log-normal Brownian motions of Merton (4, 8, 9).

I owe to Robert Merton also a seemingly paradoxical reversal of folk wisdom on age and risk-taking. This specifies a realistic case where I must rationally be less risk-taking when young (or when I am early in retirement)! It occurs when I act to maximize the expected (Exp) value of summed consumption utilities, $\Sigma U[C_i]$, over each of the time periods until retirement, of future periods ($t$) under the realistic proviso that I am wary of ever falling in consumption below each period's minimum-subistence consumption of positive $Q$. Instead of maximizing

$$\text{Exp } \Sigma U[C_i] = \text{Exp } \Sigma_i C_i^y/\gamma, \ [2]$$

I maximize

$$\text{Exp } \Sigma_i (C_i - Q)^y/\gamma, \quad 0 \neq \gamma < 1, \ Q > 0. \ [2]$$

When young, I here face for many periods the necessity of assuring the $Q$ subsistence consumptions; therefore, I must lock into safe-cash much of my portfolio when I am young. As I age and the periods left of life shrink, I need lock less of wealth in safe cash and therefore will display more tolerance for risky equities as I mature! (The earlier example of negative human capital explications and dispels the Merton paradox, just as the case of positive human capital explications this paper's new validation of folk wisdom's contention that a bequester plays safe increasingly when being older.)

In real life the quantitative importance of the desire to provide a minimum of bequest must vie with the quantitative importance of Merton's subsistence-consumption level. Therefore, simple folk wisdom is seen to be too simple, a general finding itself glimpsed by folk wisdom.


The publication costs of this article were defrayed in part by page charge payment. This article must therefore be hereby marked "advertisement" in accordance with 18 U.S.C. §1734 solely to indicate this fact.
Economic Sciences: Samuelson

REVIEW OF STANDARD MODEL

In each period a dollar of safe cash is sure to have a total return of \( r \), the safe rate of interest; \( r \geq 0 \). Each dollar in the risky security has a total return that is in each period \( t \) an independent random variable, \( X_t \), where

\[ \text{Prob}(X_t \leq x) = P(x), \quad \text{where } x \text{ is outcome per } \$ \]  \hspace{1cm} \text{(3)}

\[ P(1+r) > 0, \quad P(x) = 0 \text{ for } x < 0 \]  \hspace{1cm} \text{(4)}

\[ \text{Prob}(X_t \leq x_0, X_{t+1} \leq x_1, \ldots) = \ldots P(x_0)P(x_1) \ldots \]  \hspace{1cm} \text{(5)}

\[ \text{Exp} \{ X \} = \int_{-\infty}^{\infty} x \, dP(x) = \mu > 1 + r \]  \hspace{1cm} \text{(6)}

\[ \text{Var} \{ X \} = \int_{-\infty}^{\infty} (x - \mu)^2 \, dP(x) = \sigma^2 > 0. \]  \hspace{1cm} \text{(7)}

Under our random-walk assumptions of Eqs. 3–7, you are assumed to act like an expected (utility) maximizer who is risk-averse—as in Eq. 1 above.

Subject to the stipulations of Eqs. 1 and 3–7, you will assuredly always invest something in safe cash and something in the risky stock. Thus, when you have only one period to go, you will decide to pick your optimal fraction of portfolio in equities by the following procedure: writing \( W_t \), \( r \), \( w_t \), and \( X_t - 1 \) as your initial wealth, the interest rate, your fraction in equities, and the random-variable total returns from \$1 in risky stock, respectively, you seek

\[ \text{Max Exp} \{ U \{ [W_t(1 + r)(1 - w_t) + W_t w_t x_t] \} \} \]

\[ = \text{Exp} \{ U \{ [W_t(1 + r)(1 - w_T) + W_T w_T x_T] \} \} \]

\[ = \int_{-\infty}^{\infty} U \{ [W_t(1 + r)(1 - w_T) + W_T w_T x_T] \} \, dP(x_t) \]

\[ = U^*_t \{ W_t \}, \]  \hspace{1cm} \text{(8)}

where the optimal fraction in equities,

\[ w^*_t = f_t(\cdot) \]  \hspace{1cm} \text{(9)}

is the unique fractional root for \( w \) of

\[ 0 = \int_{-\infty}^{\infty} U^* \{ [W_t(1 + r)(1 - w) + W_t x_t] \} \times (x_t - 1 - r) \, dP(x_t). \]  \hspace{1cm} \text{(10)}

As is typical in dynamic programming, knowing \( W_t \) when there are \( T \) periods to go and having already solved recursively relations just like Eqs. 8–10 for \( U^*_t(W), U^*_t(W), \ldots, U^*_t(W) \), we can solve for our best \( w^*_t \) fraction as follows:

\[ \text{Max Exp} \{ U^*_t \{ [W_t(1 + r)(1 - w_T) + W_T w_T x_T] \} \} \]

\[ = \int_{-\infty}^{\infty} U^*_t \{ [W_t(1 + r)(1 - w_T) + W_T w_T x_T] \} \times (x_T - 1 - r) \, dP(x_T). \]  \hspace{1cm} \text{(11)}

where

\[ w^*_t = f_t(\cdot) \]  \hspace{1cm} \text{(12)}

is the unique fractional root for \( w \) of

\[ 0 = \int_{-\infty}^{\infty} U^*_t \{ [W_t(1 + r)(1 - w) + W_t x_t(x_t - 1 - r)] \} \times (x_t - 1 - r) \, dP(x_t). \]  \hspace{1cm} \text{(13)}

Folk wisdom translates into the unqualified claim that

\[ f_r(W) > f_{T-1}(W) > \ldots > f_1(W). \]  \hspace{1cm} \text{(14)}

Around 1969 many investigators deduced that this cannot be in general true:

\[ \text{for } U = \log W \text{ or } W^{r/\gamma}, \]  \hspace{1cm} \text{(15)}

all risk-averse persons follow decision rules that are "myopic"; namely, they set

\[ f_r(W) = f_{T-1}(W) = \ldots = f_1(W) = w^*_r, \]  \hspace{1cm} \text{(16)}

where \( w^*_r \) is the \( w \) root of

\[ 0 = W^r \int_{-\infty}^{\infty} [(1 + r)(1 - w) + w(x - 1 - r)]^{r-1} \times (x - 1 - r) \, dP(x). \]  \hspace{1cm} \text{(17)}

and is thus independent of \( W \) at every stage. [Remark: As \( \gamma \rightarrow 0, w^*_r \rightarrow w^*_0 \), the optimal equity fraction for the log \( W \) case. Using \( (W - 1)^{1/\gamma} \) rather than \( W^{r/\gamma} \) for \( U \) makes no difference and does let us cover the log \( W \) case by the limiting form of \( (W^{r/\gamma})^{1/\gamma} \) as \( \gamma \rightarrow 0 \).]

HANDLING SUBSISTENCE BY ESCROWED CASH

Our myopic solution for \( w^*_t \) in Eq. 17 enables us to handle the subsistence \( S \) in the \( U \) of Eq. 1. Whenever there are \( T \) periods to go, our problem is feasible only if

\[ W_T > S/(1 + r)^T. \]  \hspace{1cm} \text{(18)}

Only if we can set aside in safe cash \( S(1 + r)^{-T} \) can we ensure against the worst-case scenario where, in every period remaining, our equity’s worst-luck outcome leaves us unable to avert the infinite catastrophe of not being able to bequeath at least \( S \).

Therefore, in effect we must lock in \( S/(1 + r)^T \) into a safe-cash escrow, knowing it will grow through the stages \([S/(1 + r)^{T-1}], \ldots, S/(1 + r), S\). The remaining investable portfolio goes through the stages \([W_T, W_{T-1}, \ldots, W_0]\), where these wealths are constrained to grow exactly as indicated by the relations

\[ \tilde{W}_{T-t} = \tilde{W}_t \{ 1 + r(1 - w^*_t) + \tilde{w}_x \}, \]

\[ = W_{T-t} - S(1 + r)^{T-t}, \quad t = T, T - 1, \ldots, 1. \]  \hspace{1cm} \text{(19)}

In Eq. 19, \( \tilde{w}_x \) is the fraction of the freed-up portfolio invested in the common stock, when there are still \( t \) periods to go, and \( X_t - 1 \) is that risky security’s rate of total return.

Since our escrowed lock-in takes care of the \( S \) in our maximand’s \( U(W_0 - S) \), if we use the \( W_t \) freed-up variable, our maximand is precisely in the 1969 Santa Claus form of Eqs. 8–13, with the myopic solution of Eqs. 16 and 17:

\[ w^*_t > 0, \quad t = T, T - 1, \ldots, 2, 1. \]  \hspace{1cm} \text{(20)}

The actual portfolio, inclusive of its cash-escrowed component, displays the lower \( w^*_t \) proportion given by

\[ w^*_t = w^*_t(\tilde{W}_t/W_t) = w^*_t(W_t - S[1 + r]^{1/T})/W_t \]

\[ = \tilde{f}_t(W_t) = w^*_t(S)/(1 + r)W_t. \]  \hspace{1cm} \text{(21)}

Therefore, in agreement with folk wisdom, for \( r > 0 \) Eq. 21 does translate into

\[ \tilde{f}_t(W) > \tilde{f}_{T-1}(W) > \ldots > \tilde{f}_1(W) \]  \hspace{1cm} \text{(22)}

because of the inequality

\[ S/(1 + r)^T < S/(1 + r)^{T-1} < \ldots < S/(1 + r). \]  \hspace{1cm} \text{(23)}

Q.E.D.
THE MERTON PARADOX

Unlike the previous bequest scenario of Eq. 1, here Eq. 2’s many-period consumption scenario needs the following amount in safe-cash escrow to assure the subsistence-consumption $Q$ of each period left to go:

$$G_T = Q(1 + r)^{-1} + Q(1 + r)^{-2} + \ldots + Q(1 + r)^{-T}$$

$$= Q[1 - (1 + r)^{-T}]/r, \quad 1 + r > 0$$  \hspace{1cm} (24)

$$> G_{T-1} > G_{T-2} > \ldots > G_2 > G_1.$$  \hspace{1cm} (25)

Define the freely investable portfolio when there are $t$ periods to go by

$$\tilde{W}_t = W_t - G_t, \quad (t = T, \ldots, 2, 1).$$  \hspace{1cm} (26)

Its $\tilde{w}_t^*$ proportions held in equity are for all $t$ the constant $w_t^*$, because using the $\tilde{W}_t$s in our maximization permits us to ignore the $Q$ terms in the maximand, which already were taken care of by our declining sinking fund.

In short, the whole portfolio’s true equity fractions end up given by

$$w_t^* = w_1^*(W_t - E_t)/W_t,$$

$$= w_2^* - w_{t-1}^*(E_{t-1}/W_t) = \bar{f}_1(W_t).$$  \hspace{1cm} (27)

In view of the inequalities of Eq. 25, we verify Merton’s paradoxical increase in equity tolerance with age:

$$\bar{f}_T(W) < \bar{f}_{T-1}(W) < \ldots < \bar{f}_2(W) < \bar{f}_1(W).$$  \hspace{1cm} (28)

Q.E.D.

DISCUSSION

The present analysis dealt either with constant relative risk aversion or with subsistence-level parameters that make relative risk aversion decline from its infinite magnitude at the subsistence level to its constant level at large values of the utility function argument. (Absolute risk aversion in the Pratt–Arrow sense declines with well-being in all of our cases examined.)

Folk wisdom is shaky on just how risk aversion does change with affluence. Academic sages usually opine that (i) absolute risk tolerance rises with affluence, while (ii) relative risk tolerance may fall. The present subsistence terms, $S$ and $Q$, do not accord with the second of these speculations.

This suggests we ought to be prepared for complicated age effects, where almost anything can happen and

$$f_T(W) > f_{T-1}(W)$$  \hspace{1cm} (29)

for some $W$s and $T$ intervals.

More general than either Eq. 1 or 2 is the maximand

$$\exp(\Sigma(W_t - S_t)^\gamma/\gamma),$$

where the sequence $(S_1, \ldots, S_T)$ can be arbitrarily specified positive numbers. At any time with $t$ periods left in the planning horizon, our locked-in cash escrow has the present discounted value

$$V_t = S_1(1 + r)^{-1} + S_2(1 + r)^{-2} + \ldots + S_T(1 + r)^{-T}.$$  \hspace{1cm} (31)

By reasoning already established, we derive the most general result:

$$w_t^* > w_{t+1}^* \text{ iff } V_t > V_{t+1}$$

$$< w_{t+1}^* \text{ iff } V_t < V_{t+1}$$

$$= w_{t+1}^* \text{ iff } V_t = V_{t+1}.$$  \hspace{1cm} (32)

Finally, consider the following Merton model of instantaneous probabilities in continuous time. Denoting the passage of time by positive growth in $y = T - t$, safe cash grows like $e^{Ry}$, where $e^R = 1 + r \geq 1$. The optimized portfolio of securities obeys the following Ito–Wiener process

$$dW = [R + w(\alpha - R)]dW + wWd\alpha Z,$$

$$\alpha > R \cong 0, \quad \sigma > 0.$$  \hspace{1cm} (33)

Suppose I select $w$, over $T < t < 0$ to achieve the following maximized expected (utility):

$$[W(0) - ST]/\gamma, \quad 0 \neq \gamma < 1.$$  \hspace{1cm} (34)

It is well known from Merton (4, 8, 9) that when $S = 0$, we are in the myopic case

$$w_t^* = (\alpha - R)/\sigma^2(1 - \gamma) = w_1^*, \quad -\infty < \gamma < 1.$$  \hspace{1cm} (35)

As before we will want to set up an escrow fund $F_t$ to assure the positive $S$ subsistence need, leading to

$$\tilde{W}_t = W_t - F_t = W_t - Se^{-Rt}, \quad dF_t/dt > 0.$$  \hspace{1cm} (36)

Then by earlier arguments, we can write for positive $S$ and $R$

$$w_t^* = w_1^*[\tilde{W}_t/(W_t)] = w_1^*[1 - S/e^{Rt}(W_t) = \bar{f}_1(W_t)$$

with Eq. 36 implying that

$$\bar{f}_T(W) > \bar{f}_1(W) \text{ for } T > t > 0.$$  \hspace{1cm} (37)

The Merton paradox applies to the case where, instead of maximizing the terminal-bequest utility of Eq. 1, I am given initial $W_t$ and select $(C_t, w_t)$ to achieve the maximand of Eq. 2:

$$\text{Max} \left[ \exp \left( \gamma - \int_{0}^{T} (C_t - Q)^\gamma dt \right) \right].$$  \hspace{1cm} (39)

Now add to the coefficient of $dy$ in Eq. 33 an appendage of $-\gamma$C, with the understanding that $C$ cannot be negative. Now

$$\tilde{W}_t = W_t - E_t = W_t - Q[1 - e^{-RE}], \quad dE_t/dt > 0.$$  \hspace{1cm} (40)

$$w_t^* = \tilde{w}_1^*(\tilde{W}_t/W_t) = w_1^*[1 - E_t/W_t] = \bar{f}_1(W_t),$$

$$\bar{f}_T(W) > \bar{f}_1(W) \text{ if } T > t > 0.$$  \hspace{1cm} (41)

If we consider a third problem where we maximize a positive weighted average of the maximands of Eqs. 34 and 39, the quantitative strength of the weights has no effect on the optimal

$$w_t^* = \tilde{f}_1(W_t) = w_1^*[1 - (F_t + E_t)/W_t]$$

$$= w_1^* - w_1^*[Q - RS(e^{-Rt}/R) - (Q/R)W_t]$$  \hspace{1cm} (43)

As $S/Q$ rises from zero to a large enough magnitude, $\tilde{f}_1(W)$ changes sign.

The presence of nonrisky human capital also raises young people’s equity tolerance. However, if their wage earnings are positively correlated with stocks’ performance, this conclusion needs modifying. If wage prospects are risky but are independent of equities’ risks, Pratt (10) would point out that having certain $U(W)$ would entail my having less tolerance for one class of independent risks when another class’s riskiness rises. Folk wisdom, I fear, is not up to these subtleties: when in doubt, it perhaps simplistically expects increasing independent risks to be increasingly marginally painful?

I owe thanks to the Massachusetts Institute of Technology Center for Real Estate Development for partial postdoctoral fellowship aid; to Aase Huggins and Eva Hakala for appreciated editorial assistance; and to Robert C. Merton for stimulating discussions.