
4. The contracted tensor for a Riemann space is symmetric. Consequently if in (3.1) we replace $\Gamma^i_{jk}$ by their expressions as Christoffel symbols of the second kind for a Riemann space, the functions $\overline{\Gamma}^i_{jk}$ define an affine space possessing an invariant integral. Hence:

The spaces with paths corresponding to the paths of a Riemann space possess an invariant integral.

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CLOSED CONNECTED SETS WHICH ARE DISCONNECTED BY
THE REMOVAL OF A FINITE NUMBER OF POINTS

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Theorem A. Suppose $k$ is a positive integer and $M$ is a closed connected point set in Euclidean space of two dimensions such that

(1) if $P_1, P_2, \ldots P_k$ are any $k$ distinct points of $M$, then $M - (P_1 + P_2 + \ldots + P_k)$ is disconnected.

(2) if $Q_1, Q_2, \ldots Q_{k-1}$ are any $(k - 1)$ distinct points of $M$, then $M - (Q_1 + Q_2 \ldots Q_{k-1})$ is connected.

Under these conditions, $M$ is a continuous curve.\(^1\)

Proof. — Let us suppose that $M$ is not connected im kleinen. Then there exists a point $P$ belonging to $M$ and a circle $K$ with centre at $P$, such that within every circle whose centre is $P$ there exists a point which does not lie together with $P$ in any connected subset of $M$ that lies entirely within $K$. Let $K_1, K_2, \ldots$ denote an infinite sequence of circles with centre at $P$ and radius $r/2n$, where $r$ is the radius of $K$. Let $X_n$ denote a point within $K_n$ such that $X_n$ and $P$ do not lie together in a connected subset of $M$ which lies entirely within $K$. Let $K'$ denote a circle with centre at $P$ and radius $3r/4$. It follows with the use of a theorem due to Zoretti\(^2\) that there is a closed connected set $g_n$, containing $X_n$ and at least one point of $K'$ not containing $P$ and lying entirely within or on $K'$. It may easily be proved that there exist point sets $t_{n_1}, t_{m_1}, \ldots$ such that (1) for every $i$, $t_{n_i}$ is a closed connected subset of $M$ having at least one point on $K'$ and at least one point on $K_1$ but no point within $K_1$ or without $K'$, (2) for no values of $i$ and $j$ ($i \neq j$) does $t_{n_i}$ have a point in common with $t_{n_j}$. It follows that there exists an infinite sequence of integers $q_1, q_2, \ldots$ such that for every $i$, $q_i + 1 > q_i$ and a closed connected set $t$ and a sequence of closed connected sets $k_{n_{q_1}}, k_{n_{q_2}}, \ldots$ such that (1) for every $i$, $k_{n_{q_i}}$ is a subset of $t_{n_{q_i}}$, (2) each of
the sets \( t, k_{q_1}, k_{q_2}, \ldots \) is a subset of \((K')' - K_1\) and contains at least one point on the boundary of \( K_1\) and at least one point on the boundary of \( K'\), (3) if \( P_n q_1, P_n q_2, \ldots \) is a sequence of points such that for every \( i\), \( P_n q_i \) belongs to \( k_{q_i} \), then \( t\) contains every limit point of \( P_n q_1, P_n q_2, \ldots \).

(4) if \( q_1, q_2, q_3, \ldots \) is an infinite sequence of distinct integers belonging to the set \( q_1, q_2, \ldots \) and \( P\) is any point of \( t\), then there exists an infinite sequence of points \( P_{n q_1}, P_{n q_2}, \ldots \) such that for every \( i\), \( P_{n q_i}\) belongs to \( k_{q_i} \), and such that \( P\) is the sequential limit point of \( P_{n q_1}, P_{n q_2}, \ldots \)

Let \( P_1, P_2, \ldots, P_k\) be any \( k\) distinct points of \( t\). Then \( M - (P_1 + P_2 + \ldots + P_k) = M'_1 + M'\), two mutually separated sets.\(^3\) If \( G\) is any point of \( t\), then \( G\) is the sequential limit point of some \( G_{n q_1}, G_{n q_2}, \ldots\) such that for every \( t\), \( G_{n q_i}\) belongs to \( k_{q_i}\). Of the two sets \( M'_1\) and \( M'\), one of them, which we shall denote by \( M_1\), must contain an infinite subsequence of the point set \( G_{n q_1}, G_{n q_2}, \ldots\) But \( M_2\) denotes the other one of the sets \( M'_1\) and \( M'\). But as \( k_{q_i}\) is a connected set, \( k_{q_i}\) belongs to \( M_1\). Hence it follows that \( t - (P_1 + P_2 + \ldots + P_k)\) belongs to \( M_1\).

It follows that \( P_1 + M_2\) is connected. For suppose \( P_i + M_2 = H_1 + H_2\), two mutually separated sets and such that \( P_i\) is in \( H_1\). Then \( M - (P_i + 1 + P_i + 2 + \ldots + P_i + k - 1) = M_2 + P_i + M_1 = H_1 + H_2 + M_1\). Clearly neither of the sets \( H_1 + M_1\) and \( H_2\) contains a limit point of the other one. Thus \( H_1 + M_2\) is connected.

Let us pick out any \( k - 1\) distinct points \( P_1, P_2, \ldots, P_{k-1}\) of \( t\). Let \( G\) and \( H\) denote any two distinct points of \( t - (P_1 + P_2 + \ldots + P_{k-1})\). Then \( M - (G + P_1 + P_2 + \ldots + P_{k-1}) = S_1 + S_2\), two mutually separated set of which \( S_1\) contains \( t - (G + P_1 + P_2 + \ldots + P_{k-1})\). Likewise \( M - (H + P_1 + P_2 + \ldots + P_{k-1}) = H_1 + H_2\), two mutually separated sets of which \( H_1\) contains \( t - (H + P_1 + P_2 + \ldots + P_{k-1})\). Now \( H_1\) contains \( G\). Hence the connected set \( G + S_2\) is contained entirely in \( H_1\) while \( H_2\) is entirely in \( S_1\). If \( Q\) is any point of \( t - (P_1 + P_2 + \ldots + P_{k-1})\), then \( M - (P_1 + P_2 + P_3 + \ldots + P_{k-1} + Q) = M_1 + M_2\), two mutually separated sets. Let \( t_0\) denote that one of the sets \( M_1\) and \( M_2\) which does not contain \( t - (P_1 + P_2 + \ldots + P_{k-1} + Q)\). Let \([t_0]\) denote the set of sets thus obtained. If \( H_1\) and \( H_2\) are any two distinct points of \( t - (P_1 + P_2 + \ldots + P_{k-1})\), \( t_{H_1}\) and \( t_{H_2}\) have no points in common. We can pick out of each of the sets \([t_0]\) a definite point \( X_{t_0}\). We thus obtain a non-denumerable infinity of distinct points. One point \( B\) of \([X_{t_0}]\) must be a limit of \([X_{t_0}]\) - \( B\). Hence there exists a sequence \( B_{1}, B_{2}, B_{3}, \ldots\) of \([X_{t_0}]\) - \( B\) approaching \( B\) as its sequential limit point. Now \( B\) belongs to \( t_B\) for some point \( B'\) of \( t - (P_1 + P_2 + \ldots + P_{k-1})\) while \( B_i(i = 1, 2, 3, \ldots)\) belongs to \( t_{B_i}\) for some point \( B'_i\) of \( t - (P_1 + P_2 + \ldots + P_{k-1})\). Now \( M - (B' + P_1 + P_2 + \ldots + P_{k-1}) = M_1 + t_{B'}\). Now \( M_1\) contains \( t - (B + P_1 + P_2 + \ldots + P_{k-1})\) and hence \( B_i\) is in \( M_1\). Hence
\( B_1 + B_2 + B_3 \ldots \) is a subset of \( M_1 \). Thus \( B \) of \( t_B \) cannot be a limit point of \( B_1 + B_2 \ldots \). Hence we are led to a contradiction if we suppose \( M \) is not connected in kleinen. Hence \( M \) is a continuous curve.

2. The case where \( k = 1 \).—We shall first prove several introductory lemmas.

**Lemma A.** Suppose \( M \) is a closed connected set such that (1) \( M \) contains a point \( A \) such that \( M - A \) is connected, (2) if \( B \) is any point of \( M \) different from \( A \), then \( M - B \) is the sum of two separated sets one of which is bounded. Then \( M \) is a ray from \( A \).

**Proof.**—By methods similar to those of § 1, it follows that our set is a continuous curve. Let \( P \) denote any point of \( M \), different from \( A \). Then \( M - P = M_1 + M_2 \), two mutually separated sets of which \( M_1 \) is bounded. It follows by a theorem due to Mazurkiewicz that \( M_2 \) is unbounded. Now \( P + M_1 \) is closed and connected. We shall show that the point \( A \) is in \( M_1 \). For suppose \( A \) were in \( M_2 \). Let \( G_1 + G_2 + G_3 \ldots \) be a countable set such that every point of \( M \) either belongs to the set or is a limit point of the set. Now let \( G_{n_1} \) be the point of lowest subscript of \( G_1, G_2, \ldots \) in \( M_1 \). Let \( M - G_{n_1} = M_{11} + M_{12} \), two mutually exclusive separated sets of which \( M_{12} \) contains \( A \). It follows that \( M_2 \) is a subset of \( M_{12} \), that \( P \) is in \( M_{12} \) and \( M_{11} \) is a subset of \( M_1 \) and hence bounded. Let \( G_{n_2} \) be the point of lowest subscript of \( G_{n_{11}}, G_{n_{12}}, \ldots \) which is in \( M_1, M_2 \). Now \( M - G_{n_2} = M_{21} + M_{22} \), two mutually separated sets of which \( M_{22} \) contains \( A \). It follows that \( M_{12} \) is a subset of \( M_{22} \), \( G_{n_1} \) is in \( M_{22} \) and \( M_{21} \) is a subset of \( M_{11} \). Continue this process. For any \( i, n_i \) \( M \). The closed bounded sets \( (G_{n_1} + M_{11}), (G_{n_2} + M_{21}) \ldots \) are such that for every \( i \) \( (G_{n_{i+1}} + M_{i+1,1}) \) is a subset of \( (G_{n_i} + M_{i,1}) \). Hence they have at least one point \( X \) in common. As \( X \neq A, M - X = M' + M' \), two mutually separated sets of which \( M' \) contains \( A \). For every \( i, G_{n_i} + M_{i,2} \) must be a subset of \( M' \) while \( M_{i,1} \) contains \( M' \). Now let \( G_8 \) be the point of lowest subscript in \( M' \). Now \( G_8 \) is in \( M_{8-1,1} \) and is different from \( G_{n_8} \). Hence at this stage we did not choose the point of lowest subscript. Thus we have proved that \( A \) must be in \( M_1 \).

It may easily be proved that \( M_1 + P \) is connected and connected in kleinen. Thus there is an arc \( AP \) from \( A \) to \( P \) lying wholly on \( M_1 + P \). We shall now show that this arc \( AP \) contains the whole of \( M_1 + P \). Suppose some point \( H \) of \( M_1 \) not on the arc. Then there is on \( M_1 + P \) an arc \( HP \) from \( H \) to \( P \). Let \( T \) denote the first point of the arc \( AP \) which is an arc of \( HP \). Now as \( H \neq A, M - H = G_1 + G_2 \) two mutually separated sets of which \( G_2 \) contains \( P \). Hence the connected set \( [M_2 + P + \text{arc } AP] \) is on \( G_2 \) which is thus unbounded. But we have proved that \( A \) cannot belong to the unbounded set. Thus all points of \( M_1 + P \) are on arc \( AP \).
Let $T$ be any bounded closed connected subset of $M$ containing $A$. If $S$ is any point of $M$ not belonging to $T$, then $M - S = M_1 + M_2$, where $M_1$ contains $A$ and is bounded. It follows that $T$ belongs entirely to $M_1$. But $M_1 + S$ is an arc of $M$ from $A$ to $S$. If $P$ is any point of arc $AS - T$, then there is an arc $PS$ of $M$ free entirely of points of $T$. As any bounded closed connected subset containing more than one point of an arc is still an arc the set $T$ must be an arc from $A$ to some point $H$ of $M_1$. Now $M - H_1 = W_1 + W_2$, two mutually separated sets of which $W_1$ contains $A$. It follows that $W_1 + H_1$ is the set $T$ while $H_1$ is the only point of $T$ which is a limit point of $M - T$. Thus $M$ is a ray from $A$.

**Lemma B.** In case $k = 1$, the set $M$ of §1 is unbounded.

**Lemma C.** In case $k = 1$, the set $M$ of §1 contains no simple closed curve.

For a proof of Lemmas B and C, compare Mazurkiewicz's article.

**Theorem 1.** In case $k = 1$ and $P$ is any point of $M$, then there is an open curve $1$ of $M$ containing $P$.

**Proof.**—Suppose $M - P = M_1 + M_2$, two mutually separated sets. It follows that $P + M_i$ $(i = 1, 2)$ is a continuous curve. Hence by a theorem due to Kuratowski, there is a ray from $P$ which is a subset of $M + M_i$ $(i = 1, 2)$. The sum of these rays is an open curve through $P$.

**Theorem 2.** If $P$ is any point of $M$ then there are at most a finite number of rays $l_1, l_2 \ldots l_n$ of $M$ such that $l_i$ and $l_j$ $(i = 1, 2 \ldots n, j = 1, 2 \ldots n$ and $i \neq j)$ have no point in common other than $P$.

**Theorem 3.** Within any circle there are but a finite number of points $P_1, P_2, \ldots P_n$ which are such that at each point $P_i$ there are three or more rays of $M$ having no point in common other than $P_i$.

Theorem 2 follows immediately by methods similar to those used previously in the paper. The proof of Theorem 3, omitted here, will be given in a subsequent paper.

3. The case $k = 2$.

Suppose $k$ is an integer which is greater than or equal to $2$ and $M$ is a point set satisfying the conditions of Theorem A. Let $P_1 + P_2 \ldots P_k$ denote any $k$ distinct points of $M$. Then $M - (P_1 + P_2 \ldots P_k) = M_1 + M_2$ two mutually separated sets. It is easy to show that $M_1 + (P_1 + P_2 + \ldots + P_k)$ is connected. We shall now show that $M_1 + (P_1 + P_2 + \ldots + P_k)$ is connected in kleinian. The proof is evident if $R$ is any point of $M_1 + P_1 + P_2 \ldots P_k$ such that $R \neq P_i$ for $i = 1, 2, \ldots k$. Suppose $R = P_j$ and let $K$ be any circle having $P_j$ as centre. Put about $R$ as centre a circle $K'$ lying within $K$ and such that there is within or on $K'$ no point of the set $P_1 + P_2 \ldots + P_{j-1} + P_{j-1} + \ldots + P_k$. As $M$ is connected in kleinian there is a circle $K''$ with centre at $R$, lying within $K'$ and such that $T$ is any point of $M$ within $K''$, then $T$ and $R$ lie in a connected subset of $M$ that lies within $K'$. It may easily be proved that $T$ and $R$ are the end-points of a simple continuous arc of $M$ lying entirely within $K'$. 
Let $H$ be any point of $M_1 + P_1 + P_2 + \ldots + P_k$ in $K''$, different from $R$. Then there is an arc $R \times H$ from $H$ to $R$ belonging to $M$ and lying entirely in $K'$. From the manner in which $K'$ was chosen it follows that $R \times H - R$ is a subset of $M_1 + M_2$. Now $R \times H - R$ is connected and one point of it, $H$, lies in $M_1$. Hence as $M_1$ and $M_2$ are mutually separated, $M_2$ can contain no point of $R \times H - R$. Hence $R \times H$ is a subset of $M_1 + P_1 + P_2 + \ldots + P_k$. Thus $M_1 + P_1 + P_2 + \ldots + P_k$ is a continuous curve. In like manner $M_2 + P_1 + P_2 + \ldots + P_k$ is a continuous curve.

There is in $M_1 + P_1 + P_2 + \ldots + P_k$ ($i = 1, 2$) a simple continuous arc $P_1 X_1 P_2$. As these arcs have in common at most $k$ points and have the same end-points, it is clear that there is a simple closed curve $J$ which is a subset of $P_1 X_1 P_2 + P_1 X_2 P_3$ and thus a subset of $M$.

Let us suppose there are points of $M$ not on $J$.

There are two cases:

Case I. There is a point $T$ of $M$ within $J$. Let $A$ be any point of $J$. Then there is an arc $A \times T$ of $M$. Let $T'$ be the first point of $A \times T$ going from $T$ to $A$ which is on $J$. Take $Q_1 + Q_2 + \ldots + Q_{k-1}$, $k - 1$ distinct points of $J$ such that for every value of $i$ from 1 to $k - 1$, $Q_i \neq T'$.

Now $M = (T + Q_1 + Q_2 + \ldots + Q_{k-1}) + M_1 + M_2$, two mutually separated sets of which $M_1$ contains $T'$. As $TT' - T$ is connected all points of $TT' - T$ belong to $M_1$. Consider the set $T + Q_1 + Q_2 + \ldots + Q_{k-1} + M_2$. This set is a continuous curve and hence there is an arc $TQ_1$ lying entirely on this set. Let $Q'$ be the first point of the arc $TQ_1$ going from $T$ to $Q_1$ which is on $J$. Then arc $TT'$ and $TQ'$ is an arc of $T'TQ'$ lying except for its end-points entirely within $J$. Now there must be points of $M$ not on $J + T'TQ'$. For suppose that $M = J + T'TQ'$. There are two possibilities:

(a) $k = 2$. Let $H$ and $K$ be two points of $J$ such that $Q'$ and $T'$ separate $H$ and $K$ on $J$. Then it is clear that $J - H - K + Q'T'T'$ is connected, contrary to hypothesis.

(b) $k > 2$. Then $M - Q' - T' = Q'HT' + Q'KT' + Q'T'T'$ and hence is not connected, which is contrary to assumption.

Hence there is a point $E$ not on $J + Q'T'T'$. There are two possibilities.

(a') $E$ is without $J$. It follows there are $k - 1$ distinct points of $M$, $E_1, E_2, \ldots, E_{k-1}$ without $J$. Consider $M = (E_1 + E_2 + \ldots + E_{k-1} + T) = M_1 + M_2$, two mutually separated sets such that $T'$ is in $M_1$. It follows that all points of the connected set $J + Q'T'T' - T$ are in $M_1$. Now $T + E_1 + \ldots + E_{k-1} + M_2$ is a continuous curve. Hence there is an arc $TYE_1$ lying entirely in $T + E_1 + E_2 + \ldots + E_{k-1} + M_2$. Hence $TYE_1$ has no point in common with $J$. But as $T$ is within and $E_1$ is without $J$, the arc $TYE_1$ must have at least one point in common with $J$. Thus in case $a'$ we are led to a contradiction.

(b') $E$ is within $J$. Hence $E$ is either (i) within $HQ'T'T'H$ or (ii) within
$Q'KT'TQ'$. But in case (i) we have a point $E$ within $HQ'T'TH$ and a point $K$ without the same closed curve which is impossible by methods of case $a'$ while in case (ii) we have $E$ within $Q'KT'TQ'$ and $H$ without the same closed curve.

Thus we are led to a contradiction if we suppose a point of $M$ is within $J$. In exactly the same manner we can prove that no point of $M$ is without $J$. Hence $M = J$. But as any simple closed curve is connected by the omission of any pair of its points, then $k$ cannot be greater than 2.

Thus we are in a position to state the following theorems:

**Theorem 4.** If $M$ is a closed connected point set in Euclidean space of two dimensions such that (1) if $P$ is any point of $M$, then $M - P$ is connected, (2) if $Q$ and $R$ are any two distinct points of $M$, then $M - Q - R = M_1 + M_2$, two mutually separated sets. Under these conditions $M$ is a simple closed curve.\(^8\)

**Theorem 5.** If $K$ is any positive integer greater than 2, then there is no closed connected set $M$ satisfying the following conditions.

(a) If $P_1, P_2, \ldots, P_k$ are any $k$ distinct points of $M$ then $M - (P_1 + P_2 + \ldots + P_k)$ is disconnected.

(b) If $Q_1, Q_2, \ldots, Q_{k-1}$ are any $k - 1$ distinct points of $M$, then $M - (Q_1 + Q_2 \ldots Q_{k-1})$ is disconnected.

\(^1\) The term *continuous curve* is here used in the sense suggested by Professor R. L. Moore, who applies this term to sets which are closed, connected and connected im kleinen. Cf. R. L. Moore, *Trans. Amer. Math. Soc.*, 21 (1920), p. 347. A set of points is said to be connected im kleinen if for every point $P$ of $M$ and every circle $K$ with centre at $P$ there exists a circle $K_{h,p}$ within $K$ and with centre at $P$ such that if $X$ is a point of $M$ within $K_{h,p}$ then $X$ and $P$ lie together in some connected subset of $M$ that lies entirely within $K$. Cf. Hans Halm, *Ueber die Allgemeinsten ebene Punktmengen, die stetiges Bild einer Streichs ist*, *Jahrb. der Deut. Math. Ver.*, 23 (1914), pp. 318–22.


\(^3\) Two points sets are said to be mutually separated if neither contains a point or limit point of the other one. Cf. R. L. Moore, loc. cit., p. 341.

\(^4\) It is understood that subscripts are reduced modulo $k$.

\(^5\) An open curve is a closed and connected point set which is separated into two connected sets by the omission of any one of its points. If $P$ is a point of an open curve $M$, the point set obtained by adding $P$ to either of the two sets into which $M$ is separated by the omission of $P$ is called a ray. Prof. R. L. Moore proves that a ray is a continuous curve $M$ containing a point $A$ such that every bounded continuous subset of $M$ that contains $A$ has just one boundary point with respect to $M$. Cf. R. L. Moore, loc. cit., p. 347.


\(^8\) Compare H. Tietze, *Ueber Stetige Kurven und Jordansche Kurven*, *Math., Zts.*, 5 (1919), p. 289. Tietze makes the assumption that the set is connected im kleinen. This is really a consequence of his other conditions.