distribution of the 5 classes of build in the progeny of various matings show a difference in distribution, in means and in variability as measured by the standard deviation or the coefficient of variability, figures 4 and 5. This greater variability in the progeny of fleshy as compared with slender parents is still shown even whenever allowance is made for the slightly greater range of classes of build covered by the term fleshy, as compared with the term slender.

Finally, analysis by modern genetical methods leads to the conclusion that there are generally two or more genetic factors involved in build. There are, however, biotypes in which only one factor is involved. Markedly is this true of the biotype of medium build. The factors for fleshy build dominate slightly over those for slender build.

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A BOHR-LANGMUIR. TRANSFORMATION

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1. *Equivalence.*—Two canonical systems, with coördinates and momenta

\[ q_1, q_2, \ldots, q_k, \ p_1, p_2, \ldots, p_k; \quad Q_1, Q_2, \ldots, Q_k, \ P_1, P_2, \ldots, P_k \]

respectively, and energy functions

\[ H = E_{\text{kin}} + E_{\text{pot}}, \quad K = E'_{\text{kin}} + E'_{\text{pot}} \]

will be considered as equivalent for the purpose of this exposition, if

\[ Q_i = \Phi_i(q_1, \ldots, q_k, \ p_1, \ldots, p_k) \quad P_i = \Psi_i(q_1, \ldots, q_k, \ p_1, \ldots, p_k), \quad (1) \]

provided that the quantity \[ \Sigma(P_i \delta Q_i - p_i \delta q_i) = \delta W \quad (1') \]

can be shown by means of the equations of transformation to be the exact differential of some function \[ W(q_1, \ldots, q_k, \ p_1, \ldots, p_k). \] It is a consequence of the transformation that \[ K = H + \theta(t), \] where \( \theta(t) \) is arbitrary and may be put equal to zero without loss of generality.1

This is a principle of relativity which is inherent in the classical dynamics.

In fact, just as in the more modern theories of relativity natural coördinates and natural frames of reference are only approximately determined, so that within narrow limits a very wide range of theoretical possibility is opened, so also here there is arbitrariness in the naming of events in a dynamical process, for within certain limits of observation which
determine approximately what shall be taken for the natural positional and momental coordinates, the whole realm of change expressed by contact transformations is available. *Ipso facto*, such systems cannot be distinguished by observation. Moreover, for some systems there are no natural coordinates, i.e., none which enter into a direct relation with quantities outside the system.

1.1 Two models of the atom must be regarded as equivalent, or equally real, if each has all the energy levels of the other in the presence of the same phenomena, such as radiation, Zeeman effect, Stark effect, compression, etc. For there is then no way that one may be distinguished from another by means of these phenomena. That sort of indefiniteness is a characteristic of a quantized system. But even apart from that, it is evident that what we choose for positional and what for momental coordinates are entirely arbitrary provided that the resulting systems are equivalent by any contact transformations of the kind (1).

In this paper we investigate briefly the equivalence of the Bohr and Langmuir* single-electron atoms with respect to the phenomenon of radiation.

2. The Bohr Hydrogen Atom.—For an electron of charge $e$ and mass $m$ attracted to a fixed nucleus of charge $E$, the motion according to Bohr is given by the set of canonical equations ($i = 1, 2$)

$$
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad H = -\frac{eE}{q_i} + \frac{p_{i1}^2}{2mq_i^2} + \frac{p_{i2}^2}{2m} \tag{2}
$$

with $q_1, q_2$ respectively as the polar coordinates $\varphi, r$ in the plane of the initial motion, and $p_1 = p_\varphi = mr^2d\varphi/dt, p_2 = p_r = mdr/dt$. They are written then explicitly in the form

$$
\frac{d\varphi}{dt} = \frac{p_\varphi}{mr^2}, \quad \frac{dp_\varphi}{dt} = 0, \quad \frac{dr}{dt} = \frac{p_r}{m}, \quad \frac{dp_r}{dt} = -\frac{eE}{r^2} + \frac{p_{\varphi}^2}{mr^3} \tag{2'}
$$

with

$$
H = -\frac{eE}{r} + \frac{p_{\varphi}^2}{2mr^2} + \frac{p_r^2}{2m}.
$$

The system is further restricted in that the only stable motions are those for which

$$
\int p_i dq_i = n_i\hbar, \quad i = 1, 2, \tag{3}
$$

the integration being carried over a complete cycle of motion of the coordinate $q_i$, and $n_i$ being an integer or zero.

The particular motions for which $p_2 = 0, r = \text{const.}$ (corresponding to $n_2 = 0$) constitute the original Bohr atom with circular orbit. The more general collection of motions, in which $n_1$ and $n_2$ take on arbitrary values,
constitute the system with elliptical orbits, in which an elementary transformation allows us to take account of the motion of the nucleus, and a modification of the equations by Sommerfeld, in which the \( n_1 \) and \( n_2 \) are finally separated in the formula for energy, produces the fine structure of radiation.

3. One Type of Transformation.—As an example, we write

\[
\varphi \equiv Q_1 + P_1 + \alpha^2, \quad r = Q_2, \quad (4a)
\]

\( \alpha \) being a constant, and determine \( \varphi = q_1 \), in terms of \( Q_1, P_1 \), in such a way that \( \delta W \) shall be an exact differential. By a substitution in (1'), we calculate without difficulty

\[
\frac{\varphi}{\dot{\varphi}} = \log \left( \frac{P_1 + Q_1}{P_1 - Q_1} + f(P_1^2 - Q_1^2) \right)
\]

and take as the simplest choice \( f = \log (P_1^2 - P_2^2) \), so that

\[
\varphi = \sqrt{Q_1^2 + P_1^2 + \alpha^2} \cdot \log (P_1 + Q_1) \quad (4b)
\]

A contact transformation is thus given. Moreover

\[
K = -\frac{eE}{Q_2} + \frac{Q_1^2 + P_1^2 + \alpha^2}{2mQ_2^2} + \frac{P_2^2}{2m}
\]

and the canonical equations become

\[
\begin{align*}
\frac{dQ_1}{dt} &= \frac{\partial K}{\partial P_1} = \frac{P_1}{mQ_2^2}, \\
\frac{dQ_2}{dt} &= \frac{\partial K}{\partial P_2} = \frac{P_2}{m}
\end{align*}
\]

\[
\begin{align*}
\frac{dP_1}{dt} &= -\frac{\partial K}{\partial Q_1} = \frac{Q_1}{mQ_2^2}, \\
\frac{dP_2}{dt} &= -\frac{\partial K}{\partial Q_2} = -\frac{eE}{Q_2^2} + \frac{Q_1^2 + P_1^2 + \alpha^2}{mQ_2^3} \quad (5)
\end{align*}
\]

The first pair of equations shows us that if we interpret \( Q_1, Q_2 \) as polar coordinates \( \varphi, r \), the \( P_1, P_2 \) are the corresponding momenta in the ordinary sense, and \( m \) is the mass, unchanged. The second pair of equations describe the forces which must be assumed in order to give the motion, and the motion itself is seen to be oscillatory about a configuration determined by the interpretation of \( \alpha \). The parameter \( \alpha \) may be used as a new variable coordinate with respect to other classes of phenomena, and in this way the system may be expanded.

4. A Bohr-Langmuir Transformation.—Make now the transformation determined by the equations

\[
\varphi^2 = P^2 + \frac{n^2\hbar^2}{4\pi^2}, \quad n \text{ an integer}, \quad r = Q_2, \quad \varphi = P_2,
\]

Since again we are going to interpret \( Q_2 \) as a radial distance, we may as
well preserve the symbols $r$ and $p_r$ in the new coordinates. Substitution in (1') yields the equation 
\[ \frac{\partial}{\partial Q} \left( \frac{\varphi}{p_\varphi} \right) (P - Q) = 0. \]

Hence

\[ \frac{\varphi}{p_\varphi} P = Q + f(P), \quad f \text{ arbitrary,} \]

\[ \varphi = \sqrt{P^2 + \frac{n^2h^2}{4\pi^2}} \left\{ \frac{Q}{P} + F(P) \right\}, \quad F \text{ arbitrary,} \]

and the simplest transformation is therefore

\[ \varphi = \sqrt{P^2 + \frac{n^2h^2}{4\pi^2}} P, \quad p_\varphi = \sqrt{P^2 + \frac{n^2h^2}{4\pi^2}} \]

(6)

The inverse of this transformation, as follows from symmetry, is the following:

\[ P = \sqrt{p_\varphi^2 - \frac{n^2h^2}{4\pi^2}}, \quad Q = \frac{\varphi}{p_\varphi} \sqrt{p_\varphi^2 - \frac{n^2h^2}{4\pi^2}} \]

(6')

We shall understand $n$ to take all the values $1, 2, \ldots$, and the transformation to be defined when $p_\varphi$ lies in the neighborhood of the values $nh/2\pi$. The transformation thus defined is discontinuous.

The new system is characterized by the equations

\[ H = -\frac{eE}{r} + \frac{P^2 + \frac{n^2h^2}{4\pi^2}}{2mr^2} + \frac{p_r^2}{2m}, \quad \frac{dQ}{dt} = \frac{P}{mr^2}, \quad \frac{dr}{dt} = \frac{p_r}{m} \]

(7)

\[ \frac{dP}{dt} = 0, \quad \frac{dp_r}{dt} = -\frac{eE}{r^2} + \frac{P^2 + \frac{n^2h^2}{4\pi^2}}{mr^3} \]

The stable trajectories in (2) by the first quantum condition are by the transformation equivalent to the situations in (7) where $P = 0, Q = 0$, with $n = n_1$. The second quantum condition yields the result

\[ \int p_r dr = nh, \quad n_2 = 0, 1, 2, \ldots \]

(7')

Hence the Bohr system of stable trajectories is equivalent to the system

\[ \frac{dr}{dt} = \frac{p_r}{m}, \quad \frac{dp_r}{dt} = -\frac{eE}{r^2} + \frac{n^2h^2}{4\pi^2mr^3} \]

\[ H = -\frac{eE}{r} + \frac{n^2h^2}{8\pi^2mr^2} + \frac{p_r^2}{2m} \]

(8)

subject to condition (7'). But these are precisely the equations for the
Langmuir atom. It may be noticed that the original Bohr atom (circular orbits, \( r = \text{const.}, n_2 = 0 \)) is by this transformation equivalent to the completely static Langmuir atom, in which the possible values of \( r \) are determined by the equation

\[
0 = -\frac{E_e}{r^2} + \frac{n_i^2\hbar^2}{4\pi^2mr^3}
\]  

(9)

4.1. The transformations to which we have restricted ourselves are what may be called "natural transformations," i.e., transformations in which the energy function remains a quadratic function of the momenta, the terms involving momenta remaining homogeneous, and such that the significance of mass is unchanged. With the modern conception of the relation between mass, velocity and energy, such as we find it for instance in the Sommerfeld modification of the Bohr atom, there is no necessity for this restriction. Within the atom, mass and energy need not be regarded as separate, although we may if we like define mass at any time in terms of the coefficients of the function \( H \). For transformations in this more general situation we should want for an adequate treatment not only to introduce the \( t \) into the equations of transformation, but also to transform the parameter \( t \) itself by means of a new equation involving all the variables. The general point of view of Einsteinian relativity is however beyond the scope of this article.

5. A General Property of Contact Transformations.—It is no accident that the transformation (6) is discontinuous. In fact, the equivalence is not otherwise possible.

Consider any dynamically quantized system of \( k \) degrees of freedom which is periodic, i.e., such that for a "stable motion" the system returns to itself after a time \( \tau \):

\[ q_i(t + \tau) = q_i(t), \quad p_i(t + \tau) = p_i(t), \quad \Sigma \oint p_i dq_i = \Sigma n_i\hbar, \]  

(10)

with not all the \( n_i \) zero. Let (A) represent a contact transformation (1) in which the \( \Phi_i \) and \( \Psi_i \) are continuous functions of their arguments. Then the corresponding motion of the transformed system is also periodic with period \( \tau \) and the expression in the second of equations (10) is an integral invariant (relative) of the system; and therefore the equation

\[
\Sigma \oint P_i dQ_i = \Sigma \oint P_i dQ_i
\]  

holds for the actual motions.

A statically quantized system (one therefore with quantized forces) may be defined as one in which the stable situations satisfy the conditions \( P_i = 0, Q_i = \text{const.} \). These however cannot be the transforms of dynamically
quantized situations by any transformation \((A)\) since in that case, by \((10)\)
and \((10')\) all the \(n_i\) would have to be zero in the original system.

The same result is true even if the original or final systems, or both,
have angular coordinates, i.e., if (as in the Bohr system) we have the equations
\[
q_i(t + \tau) = q_i(t) + 2\pi, \quad i = 1, 2, \ldots, l = k,
q_i(t + \tau) = q_i(t), \quad i = l + 1, l + 2, \ldots, k,
p_i(t + \tau) = p_i(t), \quad i = 1, 2, \ldots, k.
\]

In fact, if we make the contact transformation
\[
q_i' = \sqrt{p_i} \cos q_i, \quad p_i' = -2\sqrt{p_i} \sin q_i, \quad i = 1, 2, \ldots, l, \quad (C)
q_i' = q_i, \quad p_i' = p_i, \quad i = l + 1, l + 2, \ldots, k,
\]
we obtain a system without angle coordinates, in which, as is verified
at once by the calculation of the function \(W\), the trajectories which corre-
respond to the quantized ones in the original system are again dynamically
quantized, and with the same integers:
\[
\Sigma \int p_i' dq_i' = \Sigma m_i h, \quad m_i = n_i = 1, 2, \ldots, k.
\]

If now we denote the inverse of the transformation \((C)\) by \((C^{-1})\), the
transformation from the \(q,p\) system to the \(Q,P\) system can be made by one
of the following sequences of transformations: \(CAC^{-1}, CA, AC^{-1}\), as a
result of which we are still able to deduce
\[
\Sigma \int P_i dQ_i = 0.
\]

6. Transferance Processes.—The system which we have investigated is
merely that one which is adaptable to the discussion of radiation. But
in any system where there are quantized ignorable coordinates, these may
be replaced by means of transformations \((6)\) with quantized forces, and
thus such a force may be introduced into a discussion of the Stark and
Zeeman effects.

It is of more importance to notice that any contact transformation can
itself be produced by an appropriate dynamical process. In other words,
it is always possible to construct or suppose a mechanical system whose
processes will actually carry the \(p,q\)-system into the \(P,Q\)-system, and by
assuming such a system as adjoined to the model of the atom to consider
the Bohr and Langmuir models as different states of the same atom.
Thus it is entirely legitimate for the physicist to consider the atom as a
Bohr atom for the phenomenon of radiation, and as a Langmuir atom in
the presence of chemical phenomena, the transformation from one to the
other being imagined as an actual mechanical process. Such a process may be called a transference process.

The mathematician will perhaps prefer to regard the atom merely as a sequence of events in a certain canonical process, given by equations, without regarding it as necessary to consider any given set of variables \( q_i \) as specifically configurational coordinates, and the corresponding \( p_i \) as specifically their momenta.

Summary.—The author considers the Bohr atom and the Langmuir atom with a quantized force, in regard to the question as to whether the "events" of one system are equivalent to the "events" of the other, and shows that although one system cannot be the transform of the other by any continuous contact transformation, both are however equivalent by a valid transformation which is a contact transformation locally, but not in extenso. In particular, Langmuir's completely static atom is equivalent to the Bohr atom with circular orbit.

The question of reality and preference is discussed in terms of transference processes, which carry one system into the other.

1 The more general contact transformation

\[
Q_i = \Phi_i(q_1, \ldots, q_k, p_1, \ldots, p_k; t), \quad P_i = \Psi_i(q_1, \ldots, q_k, p_1, \ldots, p_k; t)
\]

with \( W = W(q_1, \ldots, q_k, p_1, \ldots, p_k; t) \), leads to the equation

\[
K = H - (\partial W/\partial t)_{q_i, p_i} + \Theta (t).
\]

Hence energy levels will not be preserved identically for corresponding trajectories in the two systems unless \( \partial w/\partial t \) is a function of \( t \) alone. In this case, the \( \Phi_i, \Psi_i \) are independent of \( t \), and the transformation reduces to (1).

It may be remarked that any two systems of \( k \) degrees of freedom are equivalent by some contact transformation. Such a transformation will usually involve the \( t \). But even then our problem is different, since we must transform two systems one into the other in such a way that certain trajectories in one system go into certain definite trajectories in the other.


3 The canonical equations of this process may be found by constructing a Hamilton-Jacobi function \( N \) in terms of the function \( W \) of (1') and the Hamilton-Jacobi function, say \( V \), of the \( p, q \)-system. In fact, the transformation from the \( p, q \)-system with values \( q_1^0, \ldots, q_k^0, p_1^0, \ldots, p_k^0; t_0 \) to the \( P, Q \)-system with values \( Q_1, \ldots, Q_k, P_1, \ldots, P_k, t \) is itself a contact transformation with \( W \) a function of \( t \) (not quite in the sense of the footnote to Art. 1). For a non-special transformation it may be written \( W(Q_1, \ldots, Q_k, q_1^0, \ldots, q_k^0, t) \).

Introduce now, new positional coordinates \( \xi_1, \ldots, \xi_k \), and a new parameter \( \tau \) which will serve as time, taking on values from \( t_0 \) to \( t \). For the Hamilton-Jacobi function of the transference process, in terms of which the integrals of the motions of transfer are given, we may write

\[
N(\xi_1, \ldots, \xi_k, q_1^0, \ldots, q_k^0; \tau) = t - \tau \left[ V(\xi_1, \ldots, \xi_k, q_1^0; \ldots, q_k^0, t_0) + \frac{\tau - t_0}{t - t_0} W(\xi_1, \ldots, \xi_k, q_1^0, \ldots, q_k^0, t) \right]
\]