Geometric derivation of the chronometric redshift

(cosmology/Einstein universe/time evolution/discrete sources/conformal group)

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ABSTRACT The chronometric redshift–distance relation \( z = \tan(\frac{1}{2} \rho) \), where \( \rho \) is the distance in radians in the Einstein metric, is derived by an elementary geometric analysis comparable to that in traditional analysis of the expanding universe model. The differential \( dT / dT \), of Einstein time evolution \( T \), through time \( t \), as applied to the local Minkowski coordinates \( x_{\mu} \), takes the form \( \sec^{2}(\frac{1}{2} t) \). At the point of observation \( t = \rho \), implying that for a sufficiently localized source, observed wave lengths are a factor of \( \sec^{2}(\frac{1}{2} \rho) \) greater than the corresponding emitted wave lengths.

The Einstein universe (1), which was temporarily homogeneus (or “static”), fell into disfavor as a model for the large-scale universe because it appeared not to explain the cosmic redshift discovered by Slipher (2), along the lines of its early intuitive perception as a Doppler effect. But the temporarily inhomogeneous (or “dynamic”) Lorentzian manifolds proposed by de Sitter, Friedman, and Lemaître were interpretative as implicative of a Doppler spectral shift. Einstein originally strongly disputed the physical relevance of such models, which abandon the law of the conservation of energy as a well-defined cosmic principle, while granting their mathematical propriety (see, e.g., refs. 3 and 4). However, by the late 1920s, the unresolved question of the physical nature of the cosmic redshift had attained a high priority. When Hubble (5) reported the observation of an approximate local linear redshift–distance relation, as predicted by generic Doppler models of the redshift, Einstein accepted their physical tenability (see, e.g., ref. 6). On the other hand, Hubble repeatedly expressed reservations about the Doppler nature of the redshifts (see, e.g., refs. 7 and 8).

Radio astronomy led to the observation of quasars in the later 1960s (9–11), and their phenomenological rather flat magnitude–redshift relation and slowly increasing number–redshift relation ignited doubts about the validity of Friedman–Lemaître cosmology. In 1972 Segal (12) proposed a non-Doppler explanation of the redshift that was called chronometric cosmology because it involved an analysis of the concept of time. Hubble and Tolman (8) had earlier suggested, in preference to a Doppler explanation of the redshift, one that involved space curvature. It developed that the chronometric redshift was proportional to the space curvature in the Einstein universe and thus consistent with this concept. Further analysis showed that the energy loss represented by the chronometric redshift enjoyed the key features of gravitational energy, suggesting an additional interpretation in the direction of the gravitational model proposed by Zwicky (13).

Among other interesting theoretical features, chronometric cosmology restores conservation of energy to a well-defined global principle, is free of adjustable cosmological parameters such as the \( q_0 \) and \( \Lambda \) of Friedman–Lemaître cosmology, and is directly implicative of the observed redness of the cosmic shift. Its observable predictions have been consistent with objective and equitable observations on quasars, galaxies, and superluminal sources, without evolution (see, e.g., refs. 14–25). In contrast, persistent deviations of Friedman–Lemaître predictions from observation, irrespective of adjustments to the values of the two cosmological parameters, have required exculpation by whole adjustable functions representing luminosity and/or density evolution. Such evolution eliminates in principle the positive as well as the negative predictive capacity of the Doppler theory at higher redshifts. It has now been aduced not only for quasars but also for galaxies at low redshifts by Saunders et al. (26) and confirmed by statistical analysis of a model-independent sample of more than 2500 galaxies at even lower redshifts (27).

The rigorous theoretical derivation of the chronometric redshift–distance relation (28) involves the unitary representation of the conformal group on the Hilbert space of normalizable solutions of Maxwell’s equations. The traditional derivation of the redshift–distance relation in Doppler theories involves only elementary geometrical optics. The comparative sophistication of the chronometric derivation has apparently obscured the conceptual simplicity of the theory. I present here an elementary geometrical derivation of the chronometric relation, on the basis of the unicity of the covariant imbedding of Minkowski space into the Einstein universe.

Causal Symmetries of Space–Times

One-to-one causality-preserving (or conformal) transformations of Minkowski space \( M \) onto itself are necessarily products of a Poincare transformation by a dilatation (29). There is thus an 11-dimensional group \( P \) of such “causal” transformations, one dimension more than that of the isotropy group of \( M \)—i.e., the Poincare group \( P \)—itself. In the case of the Einstein universe \( E \), there is a much greater disparity in dimension between the isometry group \( K \) and the group \( G \) of all causal symmetries, the dimension of the former being 7 and that of the latter being 15. These causal symmetries act smoothly and globally on \( E \). The 8 extra dimensions can be correlated with a 7-parameter family of Poincare transformations—namely, the 4 spatio-temporal translations and the 3 Lorentz boosts—and scaling. More specifically, \( M \) can be imbedded in \( E \) in such a way that its causal symmetries extend uniquely to global causal symmetries on \( E \). In precise terms, this is the contents of

**Theorem 1.** There is a unique imbedding of \( M \) into \( E \) that:

(i) maps a given point \( O \) of \( M \) into a given point \( O' \) of \( E \) and

(ii) preserves causality (i.e., the relation of temporal precedence) and

(iii) such that for every causal symmetry \( g \) of \( M \), there is a unique causal symmetry \( g' \) of \( E \) that coincides with \( g \) on \( M \).

For the existence part of this theorem, cf. lemma 2.1.3 of ref. 30. Regarding the unicity, if \( J \) and \( J' \) are two imbeddings satisfying conditions i–iii, then \( J^{-1}J' \) is a mapping of \( M \) onto itself that preserves causality and is hence in \( P \). It follows.
from condition iii that $J^{-1}J'$ commutes with all other transformations in $P$. But the only such transformation is the identity.

With units such that $c = 1$ and $\hbar = 1$, which are used throughout, let the $x_i$ denote the usual linear coordinates in $M$, let $t$ denote the time in $E$, and let the space $S^3$ in $E$ be represented by the sphere $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$ in 4-dimensional Euclidean space. The unique imbedding $J$ of $M$ into $E$ may be conveniently expressed in terms of these quantities (30). For any point $x$ in $M$, let $A = x_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$, where the $\sigma_i$ are the Pauli matrices. For any such hermitian matrix $A$, let $U = (1 + \frac{1}{2}iA)(1 - \frac{1}{2}iA)^{-1}$. Write $U$ in the form $U = e^{i\Phi}V$, where $V$ has determinant 1 and $s$ is chosen unambiguously by continuity and agreement with the principal branch of $-\log U$ when $A$ is close to 0. Then $V$ has the form $V = u_4 + i(u_1\sigma_3 + u_2\sigma_1 + u_3\sigma_2)$, where necessarily $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$. Taking $O$ as the origin in $M$ and $O'$ as the point $t = 0$, $u_1 = u_2 = u_3 = 0$, $u_4 = 1$, $J$ carries $x$ into the point $t = s$, $u = u'$ ($j = 1, 2, 3, 4$).

The Chronometric Principle

Chronometric redshift theory is based on the premise that on the scale of discrete extragalactic sources, the universe is as originally proposed by Einstein and, in particular, is temporally homogeneous relative to Einstein time evolution. The infinitesimal generator thereof is correspondingly conserved, and it is presumed to represent the physical driving hamiltonian. Various theoretical considerations regarding the physics of extreme distances, both large and small, are suggestive of this proposal (see, e.g., refs. 31-39).

Now a free photon may be equivalently represented by a Maxwell wave function in $E$ and in $M$; every normalizable free solution of the Maxwell equations in $M$ extends uniquely to a solution in all of $E$, of the same conformally invariant norm and, conversely, in extension of the corresponding geometrical result cited above. Mathematical analysis shows that for a localized photon, the Einstein energy is not observationally distinguishable from the suitably scaled Minkowski energy. Now, there is a natural unique length scale in $E$ based on the radius of space $S^3$ as the third fundamental unit, besides $c$ and $\hbar$, required for fundamental physics (or equivalently on the minimal Einstein energy of a free photon, which is $2/R$ in units in which $\hbar = c = 1$). In $M$, however, this requisite unit is entirely conventional.

Local observational agreement of the Einstein and Minkowski energies is attainable by suitable choice of the fundamental unit of length in $M$ but is limited to the immediate vicinity of the point of observation. In particular, there is no fixed relative scale that is applicable over a cosmic time period. The Einstein and Minkowski times define the same serial order of events, but they are nonlinearly related and define quite different notions of duration on a cosmic scale. All of Minkowski time, from $-\infty$ to $+\infty$, is included in a finite Einstein time interval of duration $2\pi$. The two times cannot be synchronized throughout an historical period of cosmic duration, such as the time of flight of a photon emitted from an extragalactic source, but only at a particular instant.

The Geometrical Effect of Einstein Time Evolution

Einstein time evolution of duration $s$, say $T_s$, may be expressed in Minkowski coordinates as follows. The action of $T_s$ on $E$ is to carry the point at time $t$ and space position indicated by the matrix $V$ in $SU(2)$ into the point at time $t + s$ and at the same space position. Assuming, in order to eliminate minor complications irrelevant to the consideration of discrete sources, that $s$ is not so large that $T_s$ carries this point outside of $M$, the formalism described above is applicable. Expressing $A$ in terms of $U$ as $-2i(U - 1)(U + 1)^{-1}$, it follows that $A$ is correspondingly carried into $A(s) = -2i(e^{sU} - 1)(e^{sU} + 1)^{-1}$. This takes the following form on replacing $U$ by its expression in terms of $A$:

$$A(s) = \frac{(2a + ba)(2b - aA)}{(2b - aA)^2} = \frac{(2a + ba)(2b - aA)}{(2b - aA)^2},$$

where $a = \sin\frac{\sqrt{s}}{2}$ and $b = \cos\frac{\sqrt{s}}{2}$. Now expressing $A$ in terms of Pauli matrices and setting

$$D = 4b^2 - 4abx_0 + a^2x_0^2,$$

it follows that

$$x(s) = (2a + 2bx_0 - \frac{\sqrt{s}}{2})/2D; \quad x(s) = x_j/D.$$

Now, locally the transformation $T_s$ may be approximated arbitrarily closely in sufficiently small regions by its linearized form or differential $dT_s$. If consideration is limited to a small neighborhood of the origin in $M$, differentiation of the relations just derived shows that $dT_s$ takes the form of multiplication by $sec(\sqrt{s}/2)$, applied to all four coordinates. In particular, wave lengths after time $s$ are observed as magnified by the factor $sec^2(\sqrt{s}/2)$. Since $s = \rho$ for a free photon propagated from a distance $\rho$ at the point of observation, it appears as redshifted, in Minkowski energy, by the amount $z = tan(\frac{s}{2\rho})$.

Discussion

For correlation with a field-theoretic standpoint, it should be observed that the scaling transformation $x_\mu \rightarrow kx_\mu$ is a symmetry of Maxwell’s equations and has the effect of multiplying the energy in $M_4$ by $k^{-1}$. Note in this connection that the unitary action $U(T)$ of a geometrical transformation $T$ on a Maxwell wave function $\Psi(x)$ carries it into $m(T, x)\Psi(T^{-1}x)$, where $m(T, x)$ is the corresponding multiplier. The concept of an observed redshift implicitly assumes that the fundamental laws and constants of physics are the same in all parts of the Universe. This assumption has operational meaning only if an invariant class of observables is designated that can be compared in all parts of the universe. From an elementary particle perspective, the most basic theoretical observables are the dynamical variables that represent infinitesimal symmetries. The assumption made here is the conservative one, if the laws of physics are to be independent of position in the universe in an objective sense, that among these dynamical variables are the Einstein and Minkowski energies.

These energies $H$ and $H_0$ are well-defined self-adjoint operators in the Hilbert space $H$ of all free photon fields, in a quantum context; in a classical context, the situation is analogous, except for the replacement of Hilbert space by an infinite-dimensional symplectic manifold. $H$ contains both the emitted and observed photon wave function—there is otherwise no means to evaluate the expectation value of the same observable in both states—and is the same throughout the universe. Moreover, $H$ is the same whether space–time is represented as $E$ or as $M$, apart from nomenclature, by virtue of the unique extendability of normalizable photon fields over $M$ to the same over $E$ (40). The identity of the fundamental constants at all points of the universe is not at all directly observable on a practical time and distance scale and can only be validated by the consistency with observations of theory based on this assumption. Convenient conformally invariant units may then be chosen in chronometric cosmology so that $c = \hbar = R = 1$.

From a fundamental perspective, both the Einstein and the Minkowski energies $H$ and $H_0$ are defined globally by the corresponding infinitesimal generators of the causal symmetry group of $E$, in accordance with Theorem 1 above. The
local Minkowski coordinates $x_\mu$ near a given point are not observed in individual elementary particle observations. They appear as derived quantities that are canonically and covariantly defined by complementarity to the energy–momentum vector. Thus $x_0$ is defined at the most fundamental level as the coordinate such that $H_0$ represents the action of $\partial/\partial x_0$ on the field in question.

The treatment of the redshift given here makes essential use of the localized character of the photons emitted from discrete sources. The Hilbert space formalism is of a more fundamental character and is generally applicable. A photon of wave function $\Psi$ has the wave function $e^{-iHt}\Psi$ after time $t$ and has corresponding expected Minkowski energy $\langle H_0 e^{-iHt}\Psi, e^{-iHt}\Psi \rangle$, assuming that $\Psi$ is of unit norm. Thus, if $\Psi$ is an eigenstate of the Einstein hamiltonian, it is not redshifted at all. I thank Z. Zhou for permission to quote joint work according to which photon states with suitably re-dimensioned wave functions with $\hbar$ appearing as derived quantities covariantly defined as $\partial/\partial x_\mu$ in local Minkowski space-time, appear as derived quantities in the Hilbert space formalism.

This paper is dedicated to the memory of H. P. Robertson, in appreciation of his teaching and contributions to the development of mathematical physics.