Numerical vorticity creation based on impulse conservation

DAVID M. SUMMERS AND ALEXANDRE J. CHORIN

Mathematics Department, Napier University, 219 Colinton Road, Edinburgh, EH14 1DJ Scotland; and Department of Mathematics, University of California, Berkeley, CA 94720

Contributed by Alexandre J. Chorin, November 28, 1995

ABSTRACT The problem of creating solenoidal vortex elements to satisfy no-slip boundary conditions in Lagrangian numerical vortex methods is solved through the use of impulse elements at walls and their subsequent conversion to vortex loops. The algorithm is not uniquely defined, due to the gauge freedom in the definition of impulse; the numerically optimal choice of gauge remains to be determined. Two different choices are discussed, and an application to flow past a sphere is sketched.

In numerical methods for fluid mechanics based on Lagrangian vortex representations, the no-slip boundary condition at a solid boundary is typically satisfied by creating elements of vorticity at each time step. Although vorticity is by definition divergence-free, it has proved difficult in practice to create solenoidal numerical elements in three space dimensions, and this has been a serious source of error. Furthermore, the use of non-solenoidal elements precludes the use of powerful renormalization ("hairpin removal") strategies for simplifying calculations (1). The goal of the present paper is to show that the problem of imposing the no-slip boundary condition in a vortex method can be approached in a new and natural way, through the creation of impulse elements rather than vorticity elements. Discrete elements of impulse are also known as "magnets" in recent literature (2). Conservation of impulse (which is equivalent to conservation of momentum) makes possible the conversion of impulse elements generated at walls into closed, solenoidal vortex loops in the flow's interior. The impulse creation algorithm is not unique, because of gauge freedom in the definition of impulse. Two creation strategies will be presented; they correspond to two choices of gauge. The resulting hybrid vortex/impulse methods will be illustrated in the problem of high Reynolds number flow past a sphere.

Buttke (3) has presented a reformulation of vortex methods for flow in a region without boundaries in terms of discrete impulse elements. This reformulation trivially preserves the divergence-free character of vorticity. To apply this method to flow with boundaries, one must create impulse at walls to enforce the no-slip condition (in analogy with the creation of vorticity in standard vortex methods). Unfortunately, the use of impulse elements in the flow interior poses a number of difficulties. The strength of the elements increases very rapidly as they evolve, placing severe demands on the accuracy of the time-integration (4, 5). This problem is avoided if impulse elements are used only transitionally at walls.

Specifically, the impulse created at a wall can be assigned to closed vortex filament loops of equal impulse (6): the impulse of a vortex loop equals the product of the circulation of the loop and its spanned area. The circulation of newly created loops can be made equal for all loops. This matching of impulse allows a creation algorithm for impulse at a solid wall to be used to generate a system of vortex loops at an interface between a wall boundary-layer and the flow interior. The interior flow can then be represented by an ensemble of vortex filaments of uniform circulation that are free to advect and stretch.

The impulse density (or magnetization) \( \mathbf{m} \) is defined in terms of velocity \( \xi \) as \( \xi = \nabla \times \mathbf{u} \). The identity \( \nabla \times \mathbf{m} = \nabla \times \mathbf{u} \), where \( \mathbf{u} \) is the velocity, implies

\[
\mathbf{m} = \mathbf{u} + \nabla \phi,
\]

where \( \phi \) is a scalar function. This is a gauge freedom (2); neither \( \mathbf{m} \) nor \( \phi \) is uniquely determined. This freedom makes it possible to localize \( \mathbf{m} \)—i.e., to choose \( \phi \) so that the support of \( \mathbf{m} \) is bounded (3).

One natural generalization of familiar vortex methods (with vorticity created tangential to the boundary) is achieved by choosing \( \mathbf{m} \) perpendicular to the wall. This leads to an algorithm that embodies the creation and subsequent conservation of circulation. An alternative algorithm can be based on the choice of \( \mathbf{m} \) parallel to the boundary. At a given time step this algorithm implies the creation of an impulse density at the wall equal in magnitude to the slip-flow there, and opposite in direction; this enforces momentum conservation.

The former of these two alternatives will be illustrated here by considering the problem of flow over a sphere—i.e., flow in \( \beta \in \mathbb{R}^3 \) with an interior bounding spherical surface \( \partial \beta \).

Another application of impulse creation in a problem involving the interaction of a fluid with an elastic membrane can be found in ref. 7. A different numerical application of impulse variables is discussed in ref. 5. In view of the analysis in refs. 5 and 7, it is important to stress that impulse variables are used here to construct boundary conditions and not as a basis for a numerical solution in the flow's interior.

Flow Dynamics in Terms of Impulse

The equations of motion for incompressible viscous flow are the Navier–Stokes equations:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \mathbf{p} + \nu \Delta \mathbf{u},
\]

where \( \mathbf{u} \) is the velocity satisfying the condition \( \nabla \cdot \mathbf{u} = 0 \), and \( \mathbf{p} \) is the pressure (we assume uniform unit density, \( \rho = 1 \)). The parameter \( \nu \) is the kinematic viscosity.

The divergence of Eq. 1 yields the following:

\[
\Delta \phi = \nabla \cdot \mathbf{m}.
\]

Hence we can express the velocity in terms of the impulse as

\[
\mathbf{u} = \mathbf{m} - \nabla \Delta^{-1}(\nabla \cdot \mathbf{m}),
\]

where \( \Delta^{-1} \) represents the inverse Laplace operator.

Considering the curl of the Navier–Stokes equation (Eq. 2) and invoking vector identities and the equality \( \nabla \times \mathbf{u} = \nabla \times \mathbf{m} \), we can verify that the evolution equation for \( \mathbf{m} \),

\[
\frac{\partial \mathbf{m}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{m} + (\nabla \mathbf{u})^T \mathbf{m} = \nu \Delta \mathbf{m}, \quad \mathbf{u} = \mathbf{p} \mathbf{m},
\]

is satisfied.
is equivalent to Eq. 2 in the absence of walls. This is the gauge invariant form of Eq. 2. [In Eq. 5 the $ij$ element of the matrix $\nabla u$ is $\partial u_j/\partial x_i$, $P$ is the Hodge projection operator (8).] Multiplication of Eq. 5 by the operator $(1-P)$ yields an equation for the pressure (7):

$$p = \frac{1}{2}|u|^2 + \frac{d\phi}{dt} - \nu \Delta \phi. \quad [6]$$

The vector $u$ in a region $\Omega$ is the orthogonal projection of $m$ in $\Omega$ on the space of solenoidal vectors parallel to the boundary $\partial \Omega$. Note that $m$ is not generally solenoidal. The Hodge projection operator $P$ projects $m$ on the space of solenoidal velocities $u$.

Total impulse, a global quantity, is defined as an integral over volume:

$$M = \frac{1}{2}\int_{V} (r \times \xi)dV. \quad [7]$$

After integration by parts (4, 5), this is

$$M = \int_{\Omega} mdV.$$  

Thus $m$ can be interpreted as an impulse density.

Eq. 7 shows in particular that in the case of a single loop we have $|M| = |A_{loop}|$, where $\Gamma$ is the circulation of the loop, and $A_{loop}$ is its spanned area (see ref. 6).

In the case of inviscid flow $M$ is a conserved quantity, and therefore $dM/dt = 0$. In the case of viscous flow (where impulse is being created at a solid boundary) we can relate the time derivative of total impulse to the forces exerted by the boundary on the fluid.

The total frictional force acting on a stationary bluff body $\partial \beta$ in viscous flow is a sum of two quantities, skin friction and form drag, both of which can be expressed in terms of impulse. Skin friction equals the surface integral

$$\frac{1}{Re} \int_{\partial \beta} \nabla \cdot m \xi dS, \quad [8]$$

where $n$ is the normal to the surface. The integrand in Eq. 8 can be related to surface impulse density. Form drag is defined as

$$\int_{\partial \beta} p n dS,$$

where the pressure, $p$, is given by ref. 9. The first term in Eq. 6 (i.e., $1/2|u|^2$) can be calculated from the time derivative of the impulse associated with vorticity in the flow interior, and the contribution of the second term $(\partial \dot{\phi}/\partial t)$ can be obtained from the time derivative of the impulse associated with the potential function (10) chosen so that the impermeability condition at the surface is satisfied. In the case of a spherical surface, this potential function is determined by an ensemble of image filaments interior to the sphere.

Flow Near the Boundary

Gauge Freedom at the Wall. Standard vortex methods applied to bounded flow are based on the application of the Kelvin circulation theorem to vortex elements created at the boundary. At a given time circulation is created at a wall to extinguish the slip-field. This circulation is typically attached to a vortex sheet that participates in a particle modeling of the Prandtl boundary layer approximation (9). Upon leaving the neighborhood of the boundary, sheets are transformed into disjoint vortex segments (or blobs) bearing the circulation created at the wall. These segments are parallel to the wall and perpendicular to the slip-field at the point of creation.

To cast this boundary layer approximation in terms of impulse variables, a strategy is required to create closed vortex filament loops that are tangent to the boundary. It can be easily seen that such vorticity can be constructed from an impulse density normal to the wall because we have

$$\xi = \nabla \times m \hat{n}, \quad [9]$$

where $\hat{n}$ represents a unit normal to the wall and $\xi \cdot \hat{n} = 0$. This condition represents a particular choice of gauge.

A different choice of gauge directly expresses the conservation of momentum at the boundary: Choose $m$ parallel to the wall, and equate the impulse density at the wall to the slipp velocity there:

$$m = u_{slip}.$$  

This allows the tangential boundary forces to impart impulse to the fluid directly. Imposing this gauge into Eqs. 5 leads to equations that, formally, closely resemble vortex boundary layer equations and can be modeled in a Lagrangian fashion using sheet elements.

In our experience, the choice of gauge determines where the impulse elements (and the vortex elements that they eventually produce) are concentrated. The latter gauge ($m$ tangential) produces a more detailed picture of the boundary layer than the former gauge ($m$ normal). The best choice of gauge is likely to depend on what part of the flow one is most interested in. Analysis of the best choice of gauge remains to be done.

$m$ Normal to the Wall. We now explain in detail how one can satisfy the boundary conditions with $m$ normal to the wall, $m = (0, 0, m_z)$. One has to calculate the magnitude of $m_z$ and produce a vorticity field consistent with the velocity boundary conditions. The starting point is Eq. 9, which shows that if vorticity is tangent to the wall, then it is possible to choose an impulse density normal to the wall (11). Indeed, we can invert the relationship $\xi = \nabla \times m \hat{n}$ to determine $m_z$ on the boundary—namely,

$$m_z = \int_{0}^{x} \xi_z dx - \int_{0}^{y} \xi_y dy, \quad [10]$$

where the integration is carried out along the boundary.

A discussion in ref. 8 (pp. 74–75) is relevant here. We can choose the region in which a numerical boundary layer is confined to be sufficiently thin—specifically, $O(Re^{-\alpha})$, where $\alpha > 1/2$, with $Re$ the Reynolds number. In such a close proximity of the surface both impulse and vorticity merely diffuse. Also, we can approximate wall-vorticity by a first-order difference. Consider a boundary layer of thickness $\delta$ with an external tangential velocity field $(u_x, u_y)$. We can make the approximation $\xi = \delta^{-1}(-u_y, u_x)$. Eq. 10 becomes

$$m_z = \delta^{-1} \int_{0}^{x} u_x dx + \delta^{-1} \int_{0}^{y} u_y dy.$$  

In a more general geometry, the normal component of impulse density, $m_n$, is given by

$$m_n = \delta^{-1} \int_{C} \mathbf{u} d\mathbf{r}, \quad [11]$$

where $C$ is the element of the matrix.
with a path taken in the surface. Note that an arbitrary additive constant attaches to Eqs. 10 and 11; the choice of origin for the line-integration (Eq. 11) is immaterial.

Imagine an impulse density determined from Eq. 11 at each point of the surface $\partial \Sigma$. The surface can be partitioned into tiles (three-dimensional sheet elements), and each tile can be made to bear the appropriate local value of $m_n$. For example, this value for the $j$th tile could be determined by a numerical evaluation of

$$m_n^{(j)} = \sum_{i=1}^{j-1} \int_{x_i}^{x_{i+1}} \xi_j \, dx = \delta^{-1} \sum_{i=1}^{j-1} \int u_i \, dt,$$

(taking integration in $x$ to be generic). Each tile would then diffuse. In a random-walk representation of diffusion we would require $\delta$ to be, at most, several mean-free paths—i.e., some small multiple of $\sqrt{2dt/Re}$, with $dt$ the time-step.

During this creation process we can assign to each tile the circulation required to enforce no-slip locally; for an $h \times h$ tile, this is $h(-u_y, u_x)$ (9), following the usual vortex sheet methods. The use of a ten-function smoothing kernel leads to a smooth velocity due to sheets at the surface. The back-flow associated with vorticity is reduced, and this leads to greater numerical stability near the wall. These tiles can then be used to determine the effect of the new elements on the wall velocity as long as these elements remain within the numerical boundary layer.

**m Parallel to the Wall.** In the case of $m$ normal to the wall, the level curves of the function $m_n$ represent lines of equi-vorticity in the surface. These should form closed curves in $\partial \Sigma$ and can be themselves subpartitioned into an aggregate of closed loops conforming to the surface as suggested in refs. 2 or 12. The solenoidal property of vorticity is achieved by “closure” of the field in the plane of the wall. Determining the disposition of level curves becomes, because of this, a global exercise—specifically the line-integration in ref. 5.

The situation is different in the case of the $m$-parallel gauge. A tangential element of impulse density is equivalent to a vortex loop in a plane normal to the surface and to the direction of slip velocity. This vortex loop is closed, and the resulting algorithm is local.

**Matching Boundary to Interior**

The $m$-Normal Gauge. When tiles diffuse out of the neighborhood of the boundary, the impulse density attached to each tile can be used to determine an interim element of impulse: An “element” of $M$ is created by multiplying $m_n$ by a volume element $dV = da \, \delta$, where $da$ is the tile area. This element is then transformed into a vortex loop of equal impulse. Thus for a sheet with impulse density $m_n$, we construct a vortex loop of circulation $\Gamma$ and spanned area $A_{\text{loop}}$ such that

$$\Gamma \, A_{\text{loop}} = m_n dV. \tag{12}$$

Eq. 12 alone does not determine a vortex loop uniquely. One possibility, for example, is to create a circular vortex loop of radius $r$ oriented parallel to the surface. If the circulation of the loop is prescribed to be $\Gamma$, we can equate impulse to determine

$$\Gamma \pi r^2 = m_n \delta da.$$

This will determine the appropriate radius $r$ of the loop,

$$r = \sqrt{\frac{m_n \delta da}{\Gamma \pi}}.$$

Also we note from Eq. 11 that this loop radius can be expressed as

$$r = \left( \frac{\delta}{\Gamma \pi} \right)^{1/2}.$$

This formula emphasizes that the loop radius in this gauge does not depend explicitly on the thickness $\delta$ of the sheet layer.

In this way we can create an ensemble of interior vortex loops of equal circulation, interacting with each other by the Biot–Savart relation. The velocity at a point $x$ due to a collection of $J$ such closed filaments is given by

$$u(x) = \frac{1}{4\pi} \sum_{j=1}^{J} \frac{\delta J}{|x - x'|^3} (x - x') \times dx',$$

where $\Gamma_i$ denotes the circulation of the $i$th filament, $dx'$ is the increment of length aligned along the filament, $x'$ is the position vector of a point on the filament. The problem can be discretized in an obvious way—for example, a model introduced by Chorin (see chapter 11 in ref. 1) consists of a set of contiguous vortex tubes or segments of constant core size.

Thus Eq. 13 can be approximated by a summation of the form

$$\tilde{u}(x, t) = \frac{1}{4\pi} \sum_{j=1}^{J} \frac{\delta J}{|x - x'|^3} f(|r|/\sigma) \Gamma_j, \tag{14}$$

where $s_j$ is a segment with circulation $\Gamma_j$, $f$ is a smoothing function, and $\sigma$ is a cut-off parameter; $N$ is the number of segments representing the ensemble.

The system consists of an ensemble of polygonal loops. The vertices of these polygonal loops are to be translated in the field induced by the ensemble of segments: in general, this will result in the transport and in the stretching of a segment while preserving the connectivity of the loop. As the segment is stretched, it can be subjected to successive subdivision to preserve the initial level of discretization.

Because each segment interacts with the remaining $N - 1$ segments, this constitutes an $N$-body problem that can be handled by fast-solver strategies—e.g., the Greengard–Roklin multipole fast solver method (13). Such a strategy has been applied to ensembles of filaments (14).

The $m$-Tangential Gauge. Consider a solid boundary at $z = 0$ [where $r = (x, y, z)$ is the position vector]. The region $z > 0$ is occupied by a fluid with viscosity $\nu$ and a velocity field $u = (u_x, u_y, u_z)$. Consider also the $m$-tangential gauge—namely, $m_z = 0$. Eqs. 5 become as follows:

$$\frac{Dm_x}{Dt} = -\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} + \nu \Delta m_x,$$

$$\frac{Dm_y}{Dt} = -\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial y} + \nu \Delta m_y. \tag{15}$$

These are analogous to the equations that govern vorticity in a boundary layer and can therefore be represented in a Lagrangian sheet formulation. The sheets remain tangent to the surface. An element of created impulse density can be transported through a distance $d$ normal to the wall; this sheet object can then be made to form a vertical vortex loop with, for example, diameter $d$ by invoking conservation of impulse. If $d$ is equated to the depth of a viscous sublayer, then this scale depends on Reynolds number (this contrasts with the gauge $m$-normal, discussed previously in the $m$-normal gauge section). This vertical vortex loop is closed, and the resulting algorithm is local. The interior flow is represented by an ensemble of closed vortex filament loops interacting according to Eq. 13.

In Eq. 15 the assumption that near the wall $m$ is parallel to the wall replaces the usual boundary-layer assumptions. If,
additionally, we assume on dimensional grounds that the vorticity normal to the wall can be neglected, then Eq. 15 can be approximated by vortex sheets.

The physical meaning of this choice of gauge can be understood in the following way: A vortex sheet in a standard boundary layer approximation carries impulse: A sheet shields the fluid below it and decreases its impulse; this impulse is the strength of the sheet times the volume below it, oriented just as we described. If a sheet is removed, it must be replaced by an element of impulse with the same orientation and magnitude, and this is what is done in the tangential gauge.

One can also confirm that in two space dimensions (where an element of impulse can be viewed as a vortex dipole, and where vertical vorticity is identically zero) a special case of the analogous impulse-sheet algorithm reduces to the standard conversion of vortex sheets into vortex blobs (9). Indeed, assume that one of the vortices forming the dipole remains attached to the boundary and that the other enters the fluid; a brief calculation shows that the strength of the one that enters the fluid is exactly equal to the strength of the vortex generated by the standard algorithm; the vortex that remains at the wall is neutralized by its image. Thus, the particular choice of a vortex pair one of whose members remains at the wall produces an algorithm that is exactly equivalent to the standard conversion scheme. Note, however, that our present algorithm gives some added computational flexibility.

In all cases, note that once the impulse is ascribed to interior loops, these loops can be recombined and simplified by standard algorithms (1).

A Numerical Illustration

To illustrate the algorithm based on $m$ perpendicular to the wall we consider the flow around a sphere set impulsively into motion at time $t = 0$. Standard vortex blob methods have recently been applied to this problem (15, 16). The sphere is partitioned into quadrilateral patches. We perform a series of line-integrations in the surface of the sphere to determine the impulse density there, according to Eq. 11. In spherical coordinates $(r, \theta, \phi)$, and, with a radial impulse density, $m_r$, we use trapezoidal quadrature to evaluate

$$m_r \delta_{\theta} \theta = \int_{0}^{\phi} u \, d\ell$$

for $\phi = 0$. The slip-field $u$ is calculated as a combination of potential flow, the velocity induced by previously created sheets, and the field induced by existing filaments and their images. The slip field calculated in this way satisfies $u \cdot n = 0$. Subsequently, a series of similar integrations are then performed over $\phi \in (0, 2\pi)$,

$$m_r \delta_{\theta} \theta = \int_{0}^{\phi} u \, d\ell$$

for a set of values of $\theta \in (0, \pi)$. This generates values of $m_r \delta_{\theta}$ at the intersections of the partition mesh. A nearly uniform partition of the sphere is made using a recursively refined icosahedron, and the calculated values of $m_r$ are interpolated to centroids of the resulting triangular patches. A vortex sheet is introduced, centered at this centroid. The tangential velocity at the center of each sheet is evaluated; the calculated impulse density is simply attached to the sheet.

The sheets represent a numerical viscous sublayer (9) of thickness 6. They are subject to diffusion normal to the wall. Upon leaving this layer, each sheet is transformed into a circular vortex loop (more precisely, a polygonal approximation to a circle) in a plane tangential to $\delta \beta$ with prescribed circulation $\Gamma$. The radius of this loop is determined by matching the loss of impulse caused by the sheet’s departure from the sheet layer; for a given impulse density $m_r$ and the area $da$, we determine a loop radius $r = \sqrt{|m_r| \delta a / \Gamma}.$

To preserve impermeability at $\partial \beta$, an image loop is created simultaneously inside the sphere. This image is constructed by locating the inversion points of the polygon’s vertices. For a segment whose centroid has position vector $r$, and whose circulation is $\Gamma$, the circulation of the image is $\Gamma' = -r \cdot \Gamma$ (17). The velocity field induced by such a collection of segments and images is determined by the usual Biot–Savart algorithm of Eq. 14; we typically used a third-order Beale–Majda smoothing function and fourth-order Runge–Kutta time-integration. As segments are stretched, they are subdivided to maintain their initial level of refinement. Diffusion can be modeled in the flow interior by imparting to each loop a three-dimensional gaussian displacement; numerical experiment indicates that the contribution of this to the interior flow is small at high Reynolds number.

The presentation of the results creates severe visualization problems (as may well be expected at high Reynolds number). One can grasp the structure of the flow from multiple images or from various angles, or else one must be satisfied with partial information contained in averaged images. We have run numerical experiments up to the dimensionless time $t = 3.2$; at that time, the flow has a very plausible organization for...
the vorticity distribution in the wake. One sees an unstable (and collapsing) succession of ring-like structures shed from the sphere, resembling those described in refs. 18 and 19. Along the axial downstream direction is also an elongated braid-like vortex structure resembling qualitatively that proposed in ref. 20 on the basis of experimental observations. The coherence of the structures usually disappears in approximations that are not solenoidal, and the results illustrate the importance of the solenoidal creation algorithm. The time chosen is too short to allow comparisons of drag coefficients with experimental time-averaged values; the values we obtained are plausible. A discussion of what happens at longer integration times requires a discussion of appropriate fast summation, which we do not yet have in place. In addition, the longer the integration time, the more difficult is the visualization problem.

Visualizations of the flow at a Reynolds number of 200,000, with 1280 surface patches, at various times up to \( t = 3.2 \), are available in ref. 11 and will also be elaborated in later work. As an illustration, we exhibit in Fig. 1 one side view of the set of vortex filaments at time \( t = 3.2 \); one-sixth of the vortex loops is shown, with emphasis on the larger ones, so that the sphere itself looks denuded. Multiple images are needed to exhibit the structure of the flow. Fig. 1 should be viewed as partial evidence to the effect that the flow is very complex, the vorticity shed is prone to self-organization, numerical diffusion is not a significant factor in the calculation, and, most important in the present context, the shedding algorithm works.

This work was supported in part by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, U.S. Department of Energy under Contract DE-AC03-76SF00098.