Feynman diagrams and Wick products associated with *q*-Fock space

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It is shown that if one keeps track of crossings, Feynman diagrams can be used to compute the *q*-Wick products and usual operator products in terms of each other.

1. Introduction

A recurrent theme in noncommutative analysis is that one may use graphs to efficiently index the terms in complicated sums. One of the first to recognize this principle was Cayley (1), who introduced rooted trees in order to label differentials (see ref. 2 for additional examples). Currently, the best known example of graph-theoretic indexing may be found in perturbative quantum field theory. In this context one uses Feynman diagrams to index summands that arise when one evaluates the expectations of products of jointly Gaussian random variables (see refs. 3 and 4).

The random variables of quantum field theory correspond to certain self-adjoint operators on symmetric or antisymmetric Fock spaces (see refs. 3 and 5). In 1991, Bożejko and Speicher (6) introduced a remarkable q version of the Fock space, which for $q = 1, -1$, and 0 coincides with the symmetric (Boson), antisymmetric (Fermion), and full (Voiculescu) Fock spaces (some of these ideas had been considered in ref. 7; see also ref. 8). Bożejko and Speicher's q versions of stochastic processes and second quantization have attracted the attention of a large number of researchers (see refs. 9–11).

In ref. 8, Bożejko *et al.* introduced the q analogs of the Wick product. We show that some of the basic combinatorial calculations involving Wick products of Gaussian random variables have natural q versions. In particular, we use Feynman diagrams to express the Wick products in terms of the usual operator products and vice versa.

We will explore q forms of the Hopf algebraic theory of Kreimer (12) and Connes and Kreimer (13) in a subsequent paper.

2. *q*-Fock Spaces and Feynman Diagrams

We begin by recalling the Bożejko and Speicher (6) theory. Let H_0 be a real Hilbert space and let H be its complexification. We write

$$H^{\otimes n} = H \otimes \cdots \otimes H$$

for the algebraic tensor product, and given $-1 \leq q \leq 1$, we define a hermitian form on the algebraic sum

$$\mathcal{F}_q^{\mathrm{alg}}(H) = \mathbb{C}\Omega \oplus H \oplus H^{\otimes 2} \oplus \cdots,$$

where Ω is taken to be a unit vector, by letting

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_q = \delta_{mn} \sum_{\pi \in S_n} \langle f_1, g_{\pi(1)}\rangle \ldots \langle f_n, g_{\pi(n)}\rangle q^{\iota(\pi)},$$

and where

$$\iota(\pi) = \#\{(i,j) : 1 \leq i < j \leq n,\ \pi(i) > \pi(j)\}$$

(we use # to indicate cardinality). We will generally delete the subscript q in the hermitian form. The q-Fock space $\mathcal{F}_q(H)$ is the completion of this pre-Hilbert space. If $q = 1$ or $q = -1$, we must first divide out by the null space, and we then obtain the usual symmetric and antisymmetric Fock spaces. If $q = 0$ we obtain the full Fock space. Unless otherwise indicated, we restrict our attention to the case $-1 < q < 1$. We let $H^{[0]} = \mathbb{C}\Omega$ and

$$H^{[n]} = \mathbb{C}\Omega \oplus H \cdots \oplus H^{\otimes n}.$$

By an elementary tensor we mean an element in $H^{\otimes n}$ of the form $f_1 \otimes \cdots \otimes f_n$.

For each $f \in H_0$, the creation and annihilation operators $a^+(f)$ and $a^-(f)$ are defined on $\mathcal{F}_q^{\mathrm{alg}}(H)$ by

$$\begin{aligned} a^+(f)\Omega &= f, \\ a^+(f)(f_1 \otimes \cdots \otimes f_n) &= f \otimes f_1 \otimes \cdots \otimes f_n \end{aligned} \quad [1]$$

and

$$\begin{aligned} a^-(f)\Omega &= 0,\ a^-(f)f_1 = \langle f_1, f\rangle\Omega \\ a^-(f)(f_1 \otimes \cdots \otimes f_n) &= \sum_{i=1}^{n} q^{i-1}\langle f_i, f\rangle f_1 \\ &\quad \otimes \cdots \otimes \hat{f}_i \otimes \cdots \otimes f_n. \end{aligned} \quad [2]$$

If $q < 1$, $a^+(f)$ and $a^-(f)$ extend to bounded operators on $\mathcal{F}_q(H)$ satisfying $a^+(f) = a^-(f)^*$. They satisfy the commutation relation

$$a^-(f)a^+(g) - qa^+(g)a^-(f) = \langle f, g\rangle I.$$

We refer to the operators

$$\phi(f) = a^+(f) + a^-(f)$$

as "q Gaussians." For $q = 1$, these may be identified with jointly Gaussian random variables (see theorem I.11 in ref. 5). We define $\Gamma_q(H_0)$ to be the von Neumann algebra on $\mathcal{F}_q(H)$ generated by these operators. The vector Ω is separating and cyclic for $\Gamma_q(H_0)$. We let

$$\mathbb{E} : \Gamma_q(H_0) \to \mathbb{C} : b \mapsto \langle b\Omega, \Omega\rangle$$

be the corresponding state on $\Gamma_q(H_0)$. In particular, if $\xi = \phi(f)$ and $\eta = \phi(g)$ for $f, g \in H_0$, then we have the covariance

$$\mathbb{E}(\xi\eta) = \langle a^-(f)a^+(g)\Omega, \Omega\rangle = \langle g, f\rangle = \langle f, g\rangle$$

(we recall that H_0 is a real Hilbert space).

We wish to compute the "multivariable moments"

$$\mathbb{E}(\xi_1 \ldots \xi_m)$$

of q Gaussians $\xi_i = \phi(f_i)$, $1 \leq i \leq m$. As in the classical case, these are determined by polynomials of the "covariances" $\mathbb{E}(\xi_i\xi_j)$.

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APPLIED MATHEMATICS

We begin by letting $a^{+1}(f) = a^{+}(f)$ and $a^{-1}(f) = a^{-}(f)$. Our first task is to compute expressions of the form

$$m(\varepsilon) = \langle a^{\varepsilon(1)}(f_1)a^{\varepsilon(2)}(f_2) \cdots a^{\varepsilon(2n)}(f_{2n})\Omega, \Omega\rangle$$

for sequences $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(2n)) \in I(2n) = \{1, -1\}^{2n}$.

Given $\varepsilon \in I(2n)$, we define

$$\sigma_{2n} = \varepsilon(2n)$$

$$\sigma_{2n-1} = \varepsilon(2n-1) + \varepsilon(2n)$$

$$\cdots$$

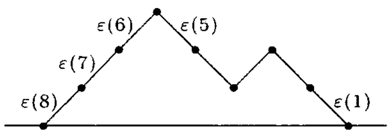
$$\sigma_2 = \varepsilon(2) + \cdots + \varepsilon(2n)$$

$$\sigma_1 = \varepsilon(1) + \varepsilon(2) + \cdots + \varepsilon(2n).$$

A simple induction shows that if $\sigma_1 > 0, \sigma_2, \ldots, \sigma_{2n} \geq 0$, then

$$a^{\varepsilon(k)}(f_k) \cdots a^{\varepsilon(2n)}(f_{2n})\Omega \in H^{[\sigma_k]}, \qquad [3]$$

and otherwise $a^{\varepsilon(k)}(f_k) \cdots a^{\varepsilon(2n)}(f_n)\Omega = 0$. We say that $\varepsilon \in I(2n)$ is a Catalan sequence if $\sigma_{2n} > 0, \sigma_{2n-1} \geq 0, \ldots, \sigma_1 = 0$, and we let $C(2n)$ be the set of such sequences. It is evident that $m(\varepsilon) = 0$ unless $\varepsilon \in C(2n)$. We may associate a Dyck path (see ref. 15, p. 221) with each $\varepsilon \in C(2n)$. For example, if $\varepsilon = (-1, -1, 1, -1, -1, 1, 1, 1) \in C(8)$, then the corresponding Dyck path is given by

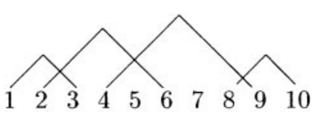


where each $\varepsilon(k)$ is the slope of the corresponding line segment. If $\varepsilon \in C(2n)$, we may evaluate $m(\varepsilon)$ in terms of "Feynman diagrams" on $[2n] = \{1, \ldots, 2n\}$ (see refs. 3 and 4 for this terminology).

Given a finite linearly ordered set S, a Feynman diagram γ on S is a partition of S into one- and two-element sets. It will be convenient to regard γ as a set of ordered pairs $\{(i_1, j_1), \ldots, (i_p, j_p)\}$ with $i_k < j_k$ and $i_k \neq i_l$ and $j_k \neq j_l$ for $k \neq l$, and we refer to the unpaired indices as "singletons." With this notation we will assume that

$$i_1 < \cdots < i_n \qquad [4]$$

(the j_k will generally be out of order). We write S^+ (respectively, S^-) for the $j \in S$ with $(k, j) \in \gamma$ for some k [respectively, the $i \in S$ such that $(i, k) \in \gamma$ for some i]. We refer to the elements of S^+ and S^- as creators and annihilators, respectively. We may specify a Feynman diagram with a simple graph of the following form



This corresponds to the Feynman diagram $\gamma = \{(1, 3), (2, 6), (4, 9), (8, 10)\}$ on $[10]$. We let $\mathsf{F}(S)$ denote the set of all Feynman diagrams on S, and we let $\mathsf{F}(n) = \mathsf{F}([n])$.

We note that more general partitions and their crossings are analyzed by using a succession of semicircles to link the elements of an equivalence class (see ref. 14).

We call the elements in S the "vertices" of the diagram. We say that a pair $(k, l) \in \gamma$ is a "left crossing" for (i, j) if $k < i < l < j$, and we define $c_l(i, j)$ to be the number of such left crossings. We refer to $c(\gamma) = \Sigma_{(i,j)\in\gamma}\, c_l(i, j)$ as the "crossing number" of γ (Biane calls this the "restricted crossing number" in ref. 14). The total left crossings can be found by counting the intersections in the corresponding graph. In the diagram shown above, $c_l(1, 3) = 0$ and $c_l(2, 6) = c_l(4, 9) = c_l(8, 10) = 1$, and thus

$$c(\gamma) = c_l(1, 3) + c_l(2, 6) + c_l(4, 9) + c_l(8, 10) = 3.$$

The general result is evident if one notes that an intersection in the graph will occur between an ascending line for one pair $(k, l) \in \gamma$ and a descending line for another $(i, j) \in \gamma$, which in turn will correspond to the left crossing (k, l) of (i, j). Similarly, $c(\gamma) = \Sigma_{(i,j)\in\gamma}\, c_r(i, j)$, where $c_r(i, j)$ is the number of right crossings $i < k < j < l$ where $(k, l) \in \gamma$.

If $(i, j) \in \gamma$, we define the "gap" $g(i, j)$ to be the number of k with $i < k < j$, and we let $a(i, j) = g(i, j) - c_l(i, j)$ (we will not need the right version of this). We define $g(\gamma) = \Sigma_{(i,j)\in\gamma}\, g(i, j)$ and $a(\gamma) = \Sigma_{(i,j)\in\gamma}\, a(i, j) = g(\gamma) - c(\gamma)$. Given $(i, j) \in \gamma$, it is evident that $c_l(i, j) + c_r(i, j) \leq g(i, j)$, and thus

$$2c(\gamma) \leq g(\gamma). \qquad [5]$$

For some purposes, it is also useful to count "degenerate crossings." These are the triples $i < k < j$, where k is not paired and $(i, j) \in \gamma$. We let $d(\gamma)$ be the number of such triples in γ, and we define the "total crossing number" to be $tc(\gamma) = c(\gamma) + d(\gamma)$.

A Feynman diagram γ on S is complete if the there are no singletons (in which case S must have an even number of elements), and we let $\mathsf{F}_c(S)$ be the collection of all such diagrams. Given $\varepsilon \in C(2n)$, we say that a complete Feynman diagram,

$$\gamma = \{(i_1, j_1), \ldots, (i_n, j_n)\},$$

on $[2n]$ is compatible with ε if $\varepsilon(i_k) = -1$ and $\varepsilon(j_k) = 1$. We let $\mathsf{F}_c(\varepsilon)$ be the set of all such Feynman diagrams on S. It is easy to see that $\mathsf{F}_c(\varepsilon)$ is nonempty. For example, we may fix a vertex of maximum height and then pair the decending and ascending edges adjacent to that vertex. In the diagram shown above we begin by placing the pair (5, 6) in γ. Eliminating the two line segments and rejoinging the graph, we can continue by induction to appropriately pair all the ascending edges with descending edges. Conversely, each complete Feynman diagram γ on $[2n]$ is compatible with the unique sequence $\varepsilon \in C(2n)$, defined by letting $\varepsilon(k) = 1$ if $k \in [2n]^+$ and $\varepsilon(k) = -1$ if $k \in [2n]^-$.

Given q-Gaussian random variables $\xi_i = \phi(f_i)$ $(i = 1, \ldots, n)$ and a Feynman diagram γ on $[n]$, we let

$$v(\gamma) = \mathbb{E}(\xi_{i_1}\xi_{j_1}) \cdots \mathbb{E}(\xi_{i_p}\xi_{j_p})\xi_{h_1} \ldots \xi_{h_r},$$

where $\gamma = \{(i_k, j_k) : k = 1, \ldots, p\}$, and $h_1 < h_2 < \cdots < h_r$ are the γ singletons.

The following is due to Bożejko and Speicher (see proposition 2 in ref. 6). We have included an alternative proof for the convenience of the reader.

Theorem 2.1. *For any* $\varepsilon \in C(2n)$,

$$m(\varepsilon) = \sum_{\gamma\in\mathsf{F}_c(\varepsilon)} v(\gamma)q^{c(\gamma)}.$$

Proof: Given $\gamma \in \mathsf{F}_c(\varepsilon)$, we will assume, as before, that $\gamma = \{(i_k, j_k)\}$, where $i_1 < \cdots < i_n$. We define a sequence of elementary tensors,

$$\Omega^{\gamma}_{2n}, \ldots, \Omega^{\gamma}_{0},$$

as follows. Let us suppose that $i_n < i_n + 1 < \cdots < j_n \leq 2n$. We define

$$\Omega^\gamma_{2n} = a^+(f_{2n})\Omega = f_{2n}$$

$$\Omega^\gamma_{2n-1} = a^+(f_{2n-1})\Omega^\gamma_{2n} = f_{2n-1}\otimes f_{2n}$$

$$\dots$$

$$\Omega^\gamma_{i_n+1} = a^+(f_{i_n+1})\Omega^\gamma_{2n-k+1} = f_{i_n+1}\otimes f_{i_n+2}\otimes\cdots\otimes f_{j_n-1}\otimes f_{j_n}\otimes\cdots.$$

From Eq. **2**,

$$a^-(f_{i_n})(\Omega^\gamma_{i_n+1}) = \langle f_{i_n}, f_{i_{n+1}}\rangle\widehat{f_{i_n+1}}\otimes f_{i_n+2}\otimes\cdots+\cdots + q^{j_n-i_n-1}\langle f_{i_n}, f_{j_n}\rangle f_{i_n+1}\otimes\cdots\otimes\hat{f}_{j_n}\otimes\cdots+\cdots.$$

We define $\Omega^\gamma_{i_n}$ to be a particular elementary tensor summand in this expression:

$$\Omega^\gamma_{i_n} = q^{j_n-i_n-1}\langle f_{i_n}, f_{j_n}\rangle f_{i_n+1}\otimes\cdots\otimes\hat{f}_{j_n}\otimes\cdots.$$

If $i_n < k < j_n$, then $k \in [2n]^+$, i.e., there is an h with $(h, k) \in \gamma$. Because $h < i_n$, it follows that (h, k) is a left crossing for (i_n, j_n), and we see that there are exactly $c(i_n, j_n) = j_n - i_n - 1$ terms between i_n and j_n. We conclude that

$$\Omega^\gamma_{i_n} = q^{c(i_n,j_n)}\langle f_{i_n}, f_{j_n}\rangle f_{i_n+1}\otimes\cdots\otimes\hat{f}_{j_n}\otimes\cdots.$$

Let us suppose that we have defined elementary tensors Ω^γ_{2n}, $\Omega^\gamma_{2n-1}, \dots, \Omega^\gamma_k$ in such a manner that $i_p < k \le i_{p+1}$, and none of the factors $f_{j_p+1}, \dots, f_{j_n}$ occur in Ω^γ_k. If $k > i_p + 1$, we let

$$\Omega^\gamma_{k-1} = a^+(f_k)\Omega^\gamma_k = f_k\otimes\Omega^\gamma_k.$$

On the other hand, if $k = i_p + 1$, let us suppose that

$$\Omega^\gamma_k = \alpha(f_{k_1}\otimes\cdots\otimes f_{k_r}\otimes f_{j_p}\otimes f_{\ell_1}\otimes\cdots)$$

for some scalar α. We define

$$\Omega^\gamma_{i_p} = \Omega^\gamma_{k-1} = q^r\alpha\langle f_{i_p}, f_{j_p}\rangle f_{k_1}\otimes\cdots\otimes f_{k_r}\otimes\hat{f}_{j_p}\otimes f_{\ell_1}\otimes\cdots,$$

which is one of the elementary tensor summands of $a^-(f_{i_p})\Omega^\gamma_{i_p+1}$. Each k_t lies in $[2n]^+\backslash\{j_{p+1}, \dots, j_n\}$ and $k_t < j_p$; hence if $k_t = j_h$, then $h < p$ and therefore $i_h < i_p$. It follows that (i_h, k_t) is a left crossing for (i_p, j_p). Because this holds for each t, we see that $r = c_\ell(i_p, j_p)$. Continuing in this manner, $\Omega^\gamma_0 = \nu(\gamma)q^{c(\gamma)}\Omega$.

Because it is evident that every nonzero elementary tensor summand in

$$a^{\varepsilon(1)}(f_1)\cdots a^{\varepsilon(2n)}(f_{2n})\Omega$$

corresponds to a unique Feynman diagram in $\mathsf{F}_c(\varepsilon)$,

$$m(\varepsilon) = \sum_{\gamma\in\mathsf{F}_c(\varepsilon)}\langle\Omega^\gamma_0, \Omega\rangle = \sum_{\gamma\in\mathsf{F}_c(\varepsilon)}\nu(\gamma)q^{c(\gamma)},$$

and we are done.

Corollary 2.1 (*q*-Wick Theorem). *For any q-Gaussian random variables ξ_i, we have*

$$\mathbb{E}(\xi_1\xi_2\dots\xi_{2n}) = \sum_{\gamma\in\mathsf{F}_c(2n)}\nu(\gamma)q^{c(\gamma)};$$

and on the other hand,

$$\mathbb{E}(\xi_1\xi_2\dots\xi_{2n+1}) = 0.$$

Proof: We have that

$$\mathbb{E}(\xi_1\dots\xi_{2n}) = \langle(a^-(f_1)+a^+(f_1))\cdots(a^-(f_{2n})+a^+(f_{2n}))\Omega, \Omega\rangle = \sum_{\varepsilon\in I(2n)} m(\varepsilon) = \sum_{\varepsilon\in C(n)} m(\varepsilon),$$

and thus the formula follows from the theorem. The result for odd moments is immediate from Eq. **3**.

3. *q*-Wick Products

Following ref. 8, we define the q-Wick product for $f_1, \dots, f_n \in H$ by

$$W(f_1, \dots, f_n) = \sum a^+(f_{i_1})\cdots a^+(f_{i_k})a^-(f_{j_1})a^-(f_{j_2})\cdots a^-(f_{j_l})q^{\iota(I,J)}, \quad [6]$$

where the sum is taken over all families $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_l\}$ where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_l$, and $I \sqcup J = [n]$, and we let $\iota(I, J) = \#(\{(p, q) : i_p > j_q\}$. This operator is characterized by the recursion

$$W(f, f_1, \dots, f_n) = \phi(f)W(f_1, \dots, f_n) - \sum_{i=1}^n q^{i-1}\langle f, f_i\rangle W(f_1, \dots, \hat{f}_i, \dots, f_n) \quad [7]$$

(see proof of proposition 2.7 in ref. 8). It follows from a simple induction on Eq. **7** that $W(f_1, \dots f_n)$ is a polynomial of the noncommuting operators $\xi_i = \phi(f_i)$, $(1 \le i \le n)$, and we will use the usual Wick product notation

$$:\xi_1\dots\xi_n: = W(f_1, \dots, f_n).$$

Because Ω is separating for $\Gamma_q(H_0)$, $u =: \xi_1 \dots \xi_n$: is the unique operator in $\Gamma_q(H_0)$ satisfying

$$u\Omega = f_1\otimes\cdots\otimes f_n.$$

We may linearly extend this to define $:p(\xi_1, \dots, \xi_n):$ for any polynomial $p(\xi_1, \dots, \xi_n)$.

The following provides an explicit (nonrecursive) expression for the q-Wick product analogous to the "classical formula" for $q = 1$ (see theorem 3.4 in ref. 4). We let $\#(\gamma)$ denote the number of pairs in γ.

Theorem 3.1. *For any q-Gaussian random variables $\xi_i = \phi(f_i)$ $(i = 1, \dots, n)$*

$$:\xi_1\cdots\xi_n: = \sum_{\gamma\in\mathsf{F}(n)}(-1)^{\#(\gamma)}q^{a(\gamma)}\nu(\gamma). \quad [8]$$

Proof: $\mathsf{F}(1)$ contains only the empty Feynman diagram γ_0, and

$$:\xi_1: = W(f_1) = a^+(f_1) + a^-(f_1) = \xi_1 = \nu(\gamma_0).$$

On the other hand, $\mathsf{F}(2) = \{\gamma_0, \gamma_1\}$, where γ_0 is empty and $\gamma_1 = \{(1, 2)\}$. We have

$$\begin{aligned}:\xi_1\xi_2: &= W(f_1, f_2) = a^+(f_1)a^+(f_2) + a^+(f_1)a^-(f_2) + qa^+(f_2)a^-(f_1) + a^-(f_1)a^-(f_2)\\ &= (a^+(f_1)+a^-(f_1))(a^+(f_2)+a^-(f_2)) - \mathbb{E}(\xi_1\xi_2)I\\ &= \xi_1\xi_2 - \mathbb{E}(\xi_1\xi_2)I = \nu(\gamma_0) - \nu(\gamma_1).\end{aligned}$$

Let us suppose that we have proved Eq. **8** for $n-1$ and n. If $\xi_0 = \phi(f_0)$, then applying the formula for n and $n-1$ and the recurrence relation,

$$W(f_0, f_1, \ldots, f_n) = \xi_0 \sum_{\gamma \in \mathsf{F}(n)} (-1)^{\#(\gamma)} q^{a(\gamma)} v(\gamma)$$

$$- \sum_{l=1}^{n} q^{l-1} \langle f_0, f_l \rangle \sum_{\delta \in \mathsf{F}([n] \setminus \{l\})} (-1)^{\#(\delta)} q^{a(\delta)} v(\delta).$$

Each Feynman diagram $\gamma = \{(i_k, j_k)\} \in \mathsf{F}(n)$ trivially determines a Feynman diagram $\gamma' = \{(i_k, j_k)\} \in \mathsf{F}(\{0\} \cup [n])$, for which $\xi_0 v(\gamma) = v(\gamma')$. Because $\#(\gamma') = \#(\gamma)$ and $a(\gamma') = a(\gamma)$,

$$\xi_0 (-1)^{\#(\gamma)} q^{a(\gamma)} v(\gamma) = (-1)^{\#(\gamma')} q^{a(\gamma')} v(\gamma').$$

Each Feynman diagram $\delta \in \mathsf{F}([n] \setminus \{l\})$ determines a Feynman diagram $\delta' = \delta \cup \{(0, l)\} \in \mathsf{F}(\{0\} \cup [n])$ for which

$$\langle f_0, f_l \rangle v(\delta) = v(\delta').$$

It is evident that $\#(\delta') = \#(\delta) + 1$. Because $c_{\delta', \ell}(0, \ell) = 0$, $a_{\delta'}(0, l) = l - 1$. If $(i, j) \in \delta$, then we may consider three cases. If $l < i$, or $j < l$, then it is evident that $a_{\delta'}(i, j) = a_\delta(i, j)$. If $i < l < j$, then $g_{\delta'}(i, j) = g_\delta(i, j) + 1$. On the other hand, $0 < i < l < j$ introduces another left crossing when we regard (i, j) as an element of δ', and thus $c_{\delta'}(i, j) = c_\delta(i, j) + 1$. It follows that $a(\delta') = l - 1 + a(\delta)$ and

$$-q^{l-1} \langle f_0, f_l \rangle (-1)^{\#(\delta)} q^{a(\delta)} v(\delta) = (-1)^{\#(\delta')} q^{a(\delta')} v(\delta').$$

A Feynman diagram θ in $\mathsf{F}(\{0\} \cup [n])$ has the form γ' if and only if 0 is a singleton in θ, and the form δ' if $(0, l) \in \theta$. Thus

$$:\xi_0 \xi_1 \cdots \xi_n: = \sum_{\theta \in \mathsf{F}(\{0\} \cup [n])} (-1)^{\#(\theta)} q^{a(\theta)} v(\theta), \qquad [9]$$

and we are done.

Conversely, we may express products $\xi_1 \ldots \xi_n$ in terms of q-Wick products. For this purpose we need to generalize the q-Wick theorem to products of q-Wick products. Given q-Gaussian random variables $\{\xi_{p,k}\}$ with $1 \le p \le t$ and $1 \le k \le n_p$, we may regard the index set $S = \{(p, k)\}$ as partitioned by the first integer, and we refer to each partition as a "block." We let S_{lex} denote S with the lexicographic ordering

$$(1, 1) < (1, 2) < \cdots (2, 1) < \cdots (t, n_t).$$

Theorem 3.2. *Suppose that we are given q-Gaussian random variables* $\{\xi_{p,k}\}$ *with* $1 \le p \le t$ *and* $1 \le k \le n_p$. *Then if* $Y_p = : \xi_{p,1} \cdots \xi_{p,n_p}:$, *we have*

$$\mathbb{E}(Y_1 \cdots Y_t) = \sum v(\gamma) q^{c(\gamma)},$$

where the sum is taken over all complete Feynman diagrams γ *on* S_{lex} *that do not link vertices within blocks.*

Proof: A typical summand of $Y_1 \cdots Y_t$ has the form

$$u = \prod_{p=1}^{t} a^+(f^{(p)}_{i_1}) \cdots a^+(f^{(p)}_{i_{r(p)}}) a^-(f^{(p)}_{j_1}) a^-(f^{(p)}_{j_2}) \ldots a^-(f^{(p)}_{j_{s(p)}}) q^{\iota(I_p, J_p)},$$

where for each p, $j_1 < j_2 < \cdots$ and $i_1 < i_2 < \cdots$.

We may use *Theorem 2.1* to compute $\langle u\Omega, \Omega \rangle$ provided we reorder the index set. We let S_u denote S with the total ordering $(p, k) < (p', k')$ if $p < p'$, and

$$(p, i_1) < (p, i_2) < \cdots < (p, j_1) < (p, j_2) < \cdots. \qquad [10]$$

If we let $\varepsilon(p, i_g) = 1$ and $\varepsilon(p, j_h) = -1$, then

$$\langle u\Omega, \Omega \rangle = m(\varepsilon) = \sum_{\gamma \in \mathsf{F}_c(\varepsilon)} v(\gamma) q^{c(\gamma)} q^{\iota(I,J)},$$

where $\iota(I, J) = \Sigma \iota(I_k, J_k)$. It should be noted that if $\gamma \in \mathsf{F}_c(\varepsilon)$, then γ will not link elements of a block, because in the definition of $v(\gamma)$ a creator is always paired with an annihilator on its left.

We may use a sequence of $\iota(I, J)$ transpositions of the index set S to transform S_u into S_{lex}. Retaining the same ordered pairs, each Feynman diagram γ on S with a given total ordering is mapped to a Feynman diagram γ' on the reordered set. It is evident that $v(\gamma) = v(\gamma')$, but in general the number of crossings will change.

If $S_u \ne S_{\text{lex}}$, then $i_{r(p)} > j_1$ for some p. Our first step will be the transition from

$$(p, i_1) < \cdots < (p, i_{r(p)}) < (p, j_1) < \cdots < (p, j_{s(p)})$$

to

$$(p, i_1) < \cdots < (p, j_1) < (p, i_{r(p)}) < \cdots < (p, j_{s(p)}).$$

Continuing in this manner, a series of $\iota(I, J)$ transpositions will give us a chain of Feynman diagrams $\gamma_1 \to \gamma_2 \to \cdots \to \gamma_{\iota(I,J)}$ on the permuted sets, which will return us to the lexicographic ordering.

At each stage we will have an adjacent "disordered" pair (p, i_k), (p, j_l) with $i_k > j_l$, and we perform the transposition

$$(p, i_k) \leftrightarrow (p, j_l).$$

Let us consider the crossings in the corresponding Feynman diagrams γ and γ'. Suppose that $a < b < c < d$ is a crossing in either γ or γ'. If none of these four vertices coincides with i_k or j_l, the crossing will be left invariant under the transpositions $\gamma \leftrightarrow \gamma'$. We have three other cases [remember that c, d and (p, j_l) are creators, and the terms (p, i_k), (p, j_l) are adjacent]:

$$a < b < (p, i_k) < c = (p, j_l) < d$$

$$a < b = (p, i_k) < (p, j_l) < c < d$$

$$a < b = (p, i_k) < c = (p, j_l) < d.$$

In the first and second cases, the crossing $a < b < c < d$ will remain unaffected under these transpositions. From our construction, the third case will occur precisely when the noncrossing sequence $a < (p, i_k) < (p, j_\ell) < d$ occurs in γ with $i_k > j_l$, and we obtain γ' by the transposition. It follows that $c(\gamma') = c(\gamma) + 1$, and thus $q^{c(\gamma)} q^h = q^{c(\gamma')} q^{h-1}$.

Starting with a Feynman diagram $\gamma = \gamma_0$ on S_u, we obtain a Feynman diagram $\gamma' = \gamma_{\iota(I,J)}$ on S_{lex} with $v(\gamma) q^{c(\gamma)} q^{\iota(I,J)} = v(\gamma') q^{c(\gamma')}$. It is easy to see that all complete diagrams on S_{lex} that do not link elements within blocks arise in this fashion; hence taking the sum of terms u, we obtain the desired result.

Theorem 3.3. *Let* $Y_i =: \xi_{i1} \cdots \xi_{in_i}:$. *Then*

$$Y_1 \cdots Y_t = \sum :v(\gamma): q^{tc(\gamma)}, \qquad [11]$$

where the sum is taken over all Feynman diagrams γ *on* S_{lex} *that do not link vertices within blocks.*

Proof: Let $A = Y_1 \cdots Y_t$ and $B = \Sigma :v(\gamma): q^{tc(\gamma)}$. As in the proof of theorem 3.15 in ref. 4, it suffices to show that for any Wick product $W =: \eta_1 \cdots \eta_u:$, $\mathbb{E}(AW) = \mathbb{E}(BW)$. From *Theorem 3.2*,

$$\mathbb{E}(AW) = \sum v(\delta) q^{c(\delta)},$$

where we sum over all complete Feynman diagrams δ on the partitioned ordered set

$$T = \{(1, 1), \ldots, (1, n_1); (2, 1), \ldots, (t, n_t); 1, \ldots, u\}$$

(where the semicolons separate blocks), which do not link vertices in the same block. Each such diagram determines a (generally incomplete) diagram $\gamma = \delta \cap (S \times S)$ on S_{lex} with the same property. Because δ is complete, the singletons $(p_1, k_1) < \ldots < (p_u, k_u)$ in γ are linked with elements of $[u] = \{1, \ldots, u\}$. It follows that

$$\nu(\delta) = \left(\prod_{(i,j)\in\gamma} \mathbb{E}(\xi_i\xi_j)\right)\mathbb{E}(\xi_{(p_1,k_1)}\eta_{g_1}) \cdots \mathbb{E}(\xi_{(p_u,k_u)}\eta_{g_u})$$
$$= \left(\prod_{(i,j)\in\gamma} \mathbb{E}(\xi_i\xi_j)\right)\nu(\theta),$$

where θ is a complete Feynman diagram on

$$T_0 = \{(p_1, k_1), \ldots, (p_u, k_u); 1, \ldots, u\}$$

that does not link vertices in the two blocks.

On the other hand, given a Feynman diagram γ on S that does not link elements in any of the blocks,

$$\nu(\gamma) = \left(\prod_{(i,j)\in\gamma} \mathbb{E}(\xi_i\xi_j)\right)\xi_{(p_1,k_1)} \cdots \xi_{(p_u,k_u)},$$

and thus

$$:\nu(\gamma): = \left(\prod_{(i,j)\in\gamma} \mathbb{E}(\xi_i\xi_j)\right):\xi_{(p_1,k_1)} \cdots \xi_{(p_u,k_u)}:.$$

From *Theorem 3.2*,

$$\mathbb{E}(:\nu(\gamma):\, W) = \left(\prod_{(i,j)\in\gamma} \mathbb{E}(\xi_i\xi_j)\right)\mathbb{E}(:\xi_{(p_1,k_1)} \cdots \xi_{(p_u,k_u)}:\, W)$$
$$= \left(\prod_{(i,j)\in\gamma} \mathbb{E}(\xi_i\xi_j)\right)\sum_{\theta\in\mathsf{G}} \nu(\theta)q^{c(\theta)},$$

where G is the set of all complete Feynman diagrams θ on T_0 that do not link vertices of the two blocks. Given such a θ, $\delta = \gamma \cup \theta$ is a generic complete Feynman diagram on T extending γ, which does not link internal vertices. It is evident that

$$c(\delta) = c(\gamma) + c(\theta) + d(\gamma),$$

because forming the term $\Pi_{(i,j)\in\gamma}\,\mathbb{E}(\xi_i\xi_j)$ will "hide" precisely $d(\gamma)$ intersections arriving from the pairs in θ. It follows that

$$\mathbb{E}(:\nu(\gamma):\, W)q^{ct(\gamma)} = \prod_{(i,j)\in\gamma} \mathbb{E}(\xi_i\xi_j)\sum_{\theta\in\mathsf{G}} \nu(\theta)q^{c(\theta)+ct(\gamma)} = \sum \nu(\delta)q^{c(\delta)},$$

where we sum over nonlinking extensions δ of γ, and thus

$$\mathbb{E}(BW) = \sum :\nu(\gamma):\, q^{tc(\gamma)} = \sum \nu(\delta)q^{c(\delta)} = \mathbb{E}(AW).$$

The one-variable case of the following result was proved by Anshelevich in remark 6.15 in ref. 11.

Corollary 3.1. *Given* $\xi_j = \phi(f_j)$ *as above, we have*

$$\xi_1 \cdots \xi_n = \sum_{\gamma\in\mathsf{F}(n)} :\nu(\gamma):\, q^{tc(\gamma)}.$$

4. The Free Case ($q = 0$) and Noncrossing Diagrams

We say that a Feynman diagram γ on a finite totally ordered set S is noncrossing if $c(\gamma) = 0$, strongly noncrossing if $tc(\gamma) = 0$, and gap-free if $g(\gamma) = 0$. We let $\mathsf{NC}(S)$, $\mathsf{SNC}(S)$, and $\mathsf{GF}(S)$ denote the corresponding diagrams on S. It is evident that

$$\mathsf{NC}(S) \supseteq \mathsf{SNC}(S) \supseteq \mathsf{GF}(S).$$

If $S = [n]$, we will simply write $\mathsf{NC}(n)$, etc.

If $q = 0$, we have the commutation relation $a^-(f)a^+(g) = \langle f, g\rangle I$. The convention of ref. 6 is that $q^c = 0$ for $c \neq 0$, and $q^0 = 1$. Thus we may drop terms in the 0-Wick theorem for which q is raised to a positive power:

$$\mathbb{E}(\xi_1\xi_2 \cdots \xi_{2n}) = \sum_{\gamma\in\mathsf{NC}(2n)} \nu(\gamma).$$

Turning to Wick products, we delete terms with inversions in the definition of the free Wick product:

$$W(f_1, \ldots, f_n) = \sum_{k=0}^{n} a^+(f_1) \cdots a^+(f_k)a^-(f_{k+1}) \cdots a^-(f_n).$$

For the alternative formula (Eq. **8**) we need only consider terms with $a(\gamma) = g(\gamma) - c(\gamma) = 0$. It follows from Eq. **5** that $g(\gamma) = 0$, and thus γ is gap-free. We conclude that

$$:\xi_1 \cdots \xi_n: = \sum_{\gamma\in\mathsf{GF}(n)} (-1)^{\#(\gamma)}\nu(\gamma). \quad \textbf{[12]}$$

We have, for example, that

$$:\xi_1\xi_2\xi_3: = \xi_1\xi_2\xi_3 - \mathbb{E}(\xi_1\xi_2)\xi_3 - \mathbb{E}(\xi_2\xi_3)\xi_1. \quad \textbf{[13]}$$

Turning to Eq. **11**, we have

$$Y_1 \cdots Y_t = \sum :\nu(\gamma):, \quad \textbf{[14]}$$

where the sum is over strongly noncrossing diagrams that do not link elements of a block. Thus, in particular,

$$\xi_1 \cdots \xi_n = \sum_{\gamma\in\mathsf{SNC}(S)} :\nu(\gamma):. \quad \textbf{[15]}$$

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1. Cayley, A. (1857) *Philos. Mag.* **13,** 172–176.
2. Brouder, C. (2000) *Eur. Phys. J. C* **12,** 521–534.
3. Glimm, J. & Jaffe, A. (1987) *Quantum Physics: A Functional Integral Point of View* (Springer, New York), 2nd Ed.
4. Janson, S. (1997) *Gaussian Hilbert Spaces* (Cambridge Univ. Press, Cambridge, U.K.), Vol. 129.
5. Simon, B. (1974) *The P(ϕ₂) Euclidean Field Theory* (Princeton Univ. Press, Princeton).
6. Bożejko, M. & Speicher, R. (1991) *Commun. Math. Phys.* **137,** 519–531.
7. Frisch, U. & Bourret, R. (1970) *J. Math. Phys.* **11,** 364–390.
8. Bożejko, M., Kümmerer, B. & Speicher, R. (1997) *Commun. Math. Phys.* **185,** 129–154.
9. Bożejko, M. & Speicher, R. (1992) in *Quantum Probability and Related Topics VII*, ed. Accardi, L. (World Scientific, Singapore), pp. 219–236.
10. Bożejko, M. & Speicher, R. (1994) *Math. Ann.* **300,** 97–120.
11. Anshelevich, M. (2001) *Doc. Math.* **6,** 343–384.
12. Kreimer, D. (1998) *Adv. Theor. Math. Phys.* **2.2,** 303–384.
13. Connes, A. & Kreimer, D. Commun. Math. Phys. (2001) *Commun. Math. Phys.* **216,** 215–241.
14. Biane, P. (1997) *Discrete Math.* **175,** 41–53.
15. Stanley, R. (1999) *Enumerative Combinatorics* (Cambridge Univ. Press, Cambridge, U.K.), Vol. 2.