Global classical solutions of the Boltzmann equation with long-range interactions

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This is a brief announcement of our recent proof of global existence and rapid decay to equilibrium of classical solutions to the Boltzmann equation without any angular cutoff, that is, for long-range interactions. We consider perturbations of the Maxwellian equilibrium states and include the physical cross-sections arising from an inverse-power intermolecular potential $r^{-(p-1)}$ with $p > 2$, and more generally. We present here a mathematical framework for unique global in time solutions for all of these potentials. We consider it remarkable that this equation, derived by Boltzmann (1) in 1872 and Maxwell (2) in 1867, grants a basic example where a range of geometric fractional derivatives occur in a physical model of the natural world. Our methods provide a new understanding of the effects due to grazing collisions.

kinetic theory | non cut-off | anisotropy | fractional derivatives

The Boltzmann equation (1) and its corresponding H-Theorem were derived by Ludwig Boltzmann in 1872. It has since become a cornerstone of statistical physics and is widely believed to accurately predict the dynamical behavior of a dilute gas. The Boltzmann equation can be written as

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = \mathcal{Q}(F,F). \quad [1]$$

The unknown $F(t,x,v)$ is a nonnegative function. For each time $t \geq 0$, $F(t,\cdot,\cdot)$ represents the density function of particles in the phase space; F may be more accurately called the empirical measure. The *Boltzmann collision operator* acts only on the velocity variables v and is local in (t,x) as

$$\mathcal{Q}(F,F)(v) = \int_{R^3} dv_* \int_{S^2} d\sigma B(|v-v_*|,\sigma)[F'_*F' - F_*F].$$

We are using the standard shorthand $F = F(v)$, $F_* = F(v_*)$, $F' = F(v')$, and $F'_* = F(v'_*)$. In this expression, v', v'_* and v, v_* are the velocities of a pair of particles before and after collision that result from parametrizing over the sphere S^2 the set of solutions to the physical law of elastic collisions:

$$v + v_* = v' + v'_* \qquad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

The *Boltzmann collision kernel* $B(|v-v_*|,\sigma)$ for a monatomic gas is, on physical grounds, a nonnegative function that only depends on the *relative velocity* $|v-v_*|$ and on the *deviation angle* θ through $\cos\theta = \langle k,\sigma\rangle$ where $k = (v-v_*)/|v-v_*|$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in R^3.

Earlier, James Clerk Maxwell (2) in 1867 showed that the collision kernels B can be computed for particles interacting according to a spherical intermolecular repulsive potential:

$$\phi(r) = r^{-(p-1)}, \quad p \in (2,+\infty). \quad [2]$$

We consider it to be a remarkable fact that this fundamental physical model, derived by Maxwell and Boltzmann in the middle of the 19th century involves geometric, nonlocal fractional differentiation effects, as we will explain in more detail below. The rigorous mathematical study of fractional Laplacians on manifolds did not begin until much later and even now there is still much to know about such operators and their role in the theory of linear and nonlinear partial differential equations.

A profound prediction of Boltzmann's equation is the Boltzmann H-Theorem*, which says that solutions satisfy

$$\frac{d}{dt}H(F) = -\frac{d}{dt}\int_{T^3_x} dx \int_{R^3_v} dvF\log F \geq 0.$$

This H-functional is therefore (formally, ignoring regularity and integrability issues) increasing and all of its maximizers are given by the Maxwellian equilibrium states:

$$\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}.$$

(This is the simplest representative of the five-parameter family of Maxwellian equilibrium states.) Boltzmann's H-Theorem demonstrates the second law of thermodynamics, which demands that the physical entropy of an isolated system does not decrease over time. It also demonstrates that Boltzmann's equation is irreversible. For statistical physics, this has been considered to be among Boltzmann's most important contributions.

Boltzmann's "proof" (1), however, and subsequent proofs of related results even up to the modern day cannot be considered completely satisfactory. The key problem is still this so-called "slight analytical difficulty," which means that we do not generally know whether or not the regularity and integrability that is required to prove the H-Theorem is propagated by the Boltzmann equation [**1**]. Although there are many useful and important fundamental theories for the Boltzmann equation, none, so far as we are aware, can be said to completely and rigorously justify the H-Theorem for the inverse-power law intermolecular potentials [**2**] with $2 < p < \infty$. We refer to refs. 1, 3–7, and the references therein in this regard.

In this work, we explain our proof of global existence of unique strong solutions to the Boltzmann equation [**1**] for the full range of intermolecular potentials [**2**]. Our solutions decay rapidly in time to the Maxwellian equilibrium states $\mu(v)$ that is the essential prediction of the Boltzmann H-Theorem. The initial data we consider are perturbations of equilibrium. Specifically, we study the linearization of [**1**] around the Maxwellian equilibrium state

$$F(t,x,v) = \mu(v) + \sqrt{\mu(v)}f(t,x,v). \quad [3]$$

Here, we consider $(t,x,v) \in [0,\infty) \times T^3_x \times R^3_x$. We suppose without restriction that the mass, momentum, and energy conservation laws for $f(t,x,v)$ hold for all $t \geq 0$ as

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*We learned from C. David Levermore and ref. 8 that Boltzmann originally called this "H" functional the "E" functional, of course, for the "Entropy," but that the typesetters of his original manuscript changed the "E" to an "H" because the German Gothic "E" was so ornate that it was hard for the untrained eye to discern.

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$$\int_{T_x^3} dx \int_{R_v^3} dv \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \sqrt{\mu(v)} f(t,x,v) = 0. \quad [4]$$

This condition should be satisfied initially, and then will continue to be satisfied for a suitably strong solution. The first statement of our main theorem is as follows:

Theorem 1. Consider the Boltzmann equation [**1**] for a spherical intermolecular repulsive potential with $p > 2$. Suppose the initial condition $f_0 = f_0(x,v)$ in [**3**] satisfies [**4**] and belongs to a suitable Hilbert space. If this initial state f_0 is sufficiently close to $\mu(v)$ then there exists a unique global in time solution to the Boltzmann equation [**1**] of the form [**3**] that decays exponentially fast to $\mu(v)$ when $p \geq 3$. If $p \in (2,3)$ then we prove decay to equilibrium with any polynomial rate. We also have positivity, i.e., $F = \mu + \sqrt{\mu} f \geq 0$ if this is so initially.

This coarsely stated main Theorem 1 is explained fully in its precise mathematical form in Theorem 2 and Theorem 3 below after the following developments.

Results and Discussion

Historical Background. As has long been known, the principal difficulty in establishing well-posedness results for the Boltzmann equation is that the singularity of the collision kernel $B(|v - v_*|, \sigma)$ in the variable $\sigma \in S^2$ is not locally integrable. In particular, this indicates that the cancellation structure of the difference $F'_* F' - F_* F$ appearing in the definition of the collision operator $\mathcal{Q}$ is crucial even at the level of definitions.

To avoid the inherent analytical difficulty of these strong singularities, Harold Grad proposed (9) in 1963 a modification of the collision operator $\mathcal{Q}$ by introducing a bounded "cutoff" near the singularity that removes these effects. Less stringent $L^1(S^2)$ cutoffs that have similar results, have also been used as early as 1954 by Morgenstern (10). These types of truncations have since been widely accepted in the mathematical literature and have now influenced several decades of progress on the Boltzmann equation. However, for the intermolecular repulsive potentials previously discussed, the cutoff theory only applies physically in the limit when $p \to \infty$ represented by Hard-Sphere particles.

Here, we give a brief collection of some breakthroughs in the context of the cutoff Boltzmann equation. In 1933, Carleman (11) proved existence and uniqueness of the spatially homogeneous problem with radial initial data. For the spatially dependent theories, it was Ukai (12) in 1974 who proved the existence of global classical solutions with close to equilibrium initial data. Ten years later, Illner–Shinbrot (13) found unique global mild solutions with near vacuum data. Then in 1989, the work of DiPerna–Lions (3) established global renormalized weak solutions for initial data without a size restriction. We also mention recent methods introduced in the linearized regime by Guo (14, 15) in 2003 and Liu-Yang-Yu (16) in 2004. We do use the space-time estimates from ref. 14 in our proof. We refer to ref. 5 for a long review of mathematical Kinetic theory.

The Grad angular cutoff assumption was originally believed to not change the essential nature of solutions to the equation. Historically, it has been furthermore argued that, physically, the important properties of the Boltzmann equation are not particularly sensitive to the dependence of the collision kernel upon the deviation angle: θ (5).

There is, however, a long history of mathematical results at the mesoscopic level of Boltzmann, which illustrate that, instead, solutions of the Boltzmann equation have strong dependence on the angular singularity either locally in time, or in the context of renormalized weak solutions or for initial data without any spatial dependence; see refs. 5, 7, 17–20, and the references therein. In particular, the large data theory of global renormalized weak solutions, due to DiPerna-Lions (3), has been established for the Boltzmann equation without cutoff in the paper by Alexandre–Villani (7) from 2002 if one adds to the equation a nonnegative defect measure that is at the present time not known to vanish. Our results additionally show that, for all of the intermolecular repulsive potentials [**2**], our solutions exhibit a gain of velocity regularity globally in time.

It was shown by Boudin–Desvillettes (21) in 2000 that under the angular cutoff assumption solutions near vacuum have the same regularity in a Sobolev space as the initial data. On the other hand, when the physical effects of the angular singularity are not cutoff the Boltzmann equation is well-known to experience smoothing effects. These results go back to Lions (22) and Desvillettes (23) and have seen substantial developments. Recently, Chen–Desvillettes–He (24) and also Alexandre–Morimoto–Ukai–Xu–Yang (20) have developed independent machinery to study these general smoothing effects for kinetic equations. These results illustrate that the Boltzmann equation with angular cutoff is in some respects very different from the one without any angular cutoff.

Moreover, the Boltzmann collision operator without cutoff has been widely conjectured to "behave" essentially as the fractional flat diffusion $-(-\Delta_v)^s$. This can be expressed as

$$F \mapsto Q(g,F) \sim -(-\Delta_v)^s F + \text{l.o.t.}$$

Above "l.o.t." indicates that the remaining terms will be lower order. The original mathematical intuition for this conjecture goes back to Carlo Cercignani (25) in 1969 over forty years ago (5). This conjecture has been shown to be correct in terms of the smoothing effects induced by the Boltzmann collision operator as has been proven, for instance, in refs. 20 and 22–24. Even so, our research proves at the linearized level that the essential behavior of the collision operator is not quite a flat fractional diffusion. Instead, there are directionally dependent velocity weights that go to infinity as velocity goes to infinity that are crucially intertwined with the fractional diffusive effects. We are able to characterize this interconnection "geometrically." In fact, the linear behavior is exactly that of a fundamentally nonisotropic fractional Laplacian with the geometry of a "lifted" paraboloid in R^4. We illustrate entropy production estimates in the last section that demonstrates the same nonisotropic fractional diffusive effects for the Boltzmann "entropy production functional" in the fully nonlinear context.

Formulation. The linearization [**3**] of the Boltzmann equation [**1**] grants an equation for the perturbation $f(t,x,v)$ as

$$\partial_t f + v \cdot \nabla_x f + L(f) = \Gamma(f,f), \qquad f(0,x,v) = f_0(x,v), \quad [5]$$

where the *linearized Boltzmann operator* L is given by

$$\begin{aligned} L(g) &\overset{\text{def}}{=} -\mu^{-1/2}\mathcal{Q}(\mu, \sqrt{\mu} g) - \mu^{-1/2}\mathcal{Q}(\sqrt{\mu} g, \mu) \\ &= \int_{R^3} dv_* \int_{S^2} d\sigma B(|v - v_*|, \cos\theta) \\ &\quad \times [g_* M + g M_* - g'_* M' - g' M'_*] M_*, \end{aligned}$$

and the bilinear operator Γ takes the form

$$\Gamma(g,h) \overset{\text{def}}{=} \mu^{-1/2}\mathcal{Q}(\sqrt{\mu} g, \sqrt{\mu} h) = \int_{R^3} dv_* \int_{S^2} d\sigma B M_* (g'_* h' - g_* h).$$

In both definitions, $M(v) \overset{\text{def}}{=} \sqrt{\mu} = (2\pi)^{-3/4} e^{-|v|^2/4}$, and we use the parametrization for collisional variables as

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$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \qquad \sigma \in S^2.$$

We further decompose $L = N + K$. Here, N is the "norm part" of the linearized operator and K will be seen as the "compact part." The norm part is then written as

$$Ng \stackrel{\text{def}}{=} -\Gamma(M,g) - \nu_K(v)g = -\int_{R^3} dv_* \int_{S^2} d\sigma B(g'-g)M'_* M_* + \nu(v)g(v),$$

where the weight $\nu_K(v)$ is suitably chosen (depending on various technical considerations) so that $\nu(v)$ is nonnegative and $\nu_K(v) \ll \nu(v)$ as $|v| \to \infty$. We will use $\langle \cdot, \cdot \rangle$ to denote the standard $L^2(R^3_v)$ inner product. With this notation, the norm piece may easily be shown to satisfy the following identity:

$$\langle Ng, g \rangle = \frac{1}{2}\int_{R^3} dv \int_{R^3} dv_* \int_{S^2} d\sigma B(g'-g)^2 M'_* M_* + \int_{R^3} dv \nu(v)|g(v)|^2.$$

We also record here the definition of the "compact part" K:

$$Kg \stackrel{\text{def}}{=} \nu_K(v)g - \Gamma(g,M) = \nu_K(v)g - \int_{R^3} dv_* \int_{S^2} d\sigma B M_*(g'_* M' - g_* M).$$

This is our main splitting of the linearized operator.

As will be seen, our solution to the problem at hand rests heavily on our introduction of the following weighted geometric fractional Sobolev space:

$$N^{s,\gamma} \stackrel{\text{def}}{=} \{g \in L^2(R^3_v) : |g|_{N^{s,\gamma}} < \infty\},$$

where the norm $|g|^2_{N^{s,\gamma}}$ is given by the formula

$$\int_{R^3} dv \int_{R^3} dv' \frac{(g'-g)^2}{d(v,v')^{3+2s}} (\langle v \rangle \langle v' \rangle)^{\frac{\gamma+2s+1}{2}} \mathbf{1}_{d(v,v') \le 1} + |g|^2_{L^2_{\gamma+2s}}.$$

In this formula, $\langle v \rangle \stackrel{\text{def}}{=} \sqrt{1+|v|^2}$ and the weighted-L^2 space,

$$|g|^2_{L^2_{\gamma+2s}} \stackrel{\text{def}}{=} \int_{R^3} dv \langle v \rangle^{\gamma+2s} |g(v)|^2,$$

are as usual. The metric term $d(v,v')$ is given by

$$d(v,v') \stackrel{\text{def}}{=} \sqrt{|v-v'|^2 + \left(\frac{1}{2}|v|^2 - \frac{1}{2}|v'|^2\right)^2}.$$

The inclusion of the quadratic difference $|v|^2 - |v'|^2$ is an essential part of the linearized Boltzmann collision operator; it is not a lower order term. In particular, if for every $v \in R^3$ one defines $\underline{v} \stackrel{\text{def}}{=} (v, \frac{1}{2}|v|^2) \in R^4$, then $d(v,v')$ equals the distance in R^4 between $\underline{v}$ and $\underline{v'}$. Thus, the presence of this term indicates that the underlying geometry is that of a paraboloid rather than flat space. In fact our $N^{s,\gamma}$ norm is sharp for the linearized Boltzmann collision operator, in particular $|g|^2_{N^{s,\gamma}}$ and $\langle Ng, g \rangle$ are equivalent.

To state our main result precisely, we must first give a few additional details about the nature of the collision kernel B. The assumptions made below and in refs. 26 and 27 are as follows:

- $B(|v-v_*|, \sigma)$ takes product form in its arguments as
$$B(|v-v_*|, \sigma) = \Phi(|v-v_*|) b(\cos\theta).$$
In general, both b and Φ are nonnegative functions.
- For fixed $s \in (0,1)$, $c_b > 0$, the angular factor satisfies
$$\frac{c_b}{\theta^{1+2s}} \le \sin\theta b(\cos\theta) \le \frac{1}{c_b \theta^{1+2s}}, \quad \forall\, \theta \in \left(0, \frac{\pi}{2}\right].$$
- The kinetic factor $z \mapsto \Phi(|z|)$ satisfies for some $C_\Phi > 0$
$$\Phi(|v-v_*|) = C_\Phi |v-v_*|^\gamma, \qquad \gamma > -2s-1.$$

These conditions hold for all of the collision kernels that arise from the intermolecular potential [2] with $p > 2$.

We will work in the Hilbert spaces $L^2_v H^K_x(T^3_x \times R^3_v)$ where

$$\|f\|^2_{L^2_v H^K_x} \stackrel{\text{def}}{=} \sum_{|\alpha| \le K} \int_{R^3_v} dv \int_{T^3_x} dx |\partial^\alpha_x f(x,v)|^2.$$

This is the Sobolev space with $\partial^\alpha_x = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3}$. We have

Theorem 2. Suppose $\gamma + 2s \ge 0$ and fix $K \ge 3$. Choose the initial state $f_0 = f_0(x,v) \in L^2_v H^K_x(T^3_x \times R^3_v)$ in [**3**] that satisfies [**4**]. There is an $\eta_0 > 0$ such that if $\|f_0\|_{L^2_v H^K_x} \le \eta_0$ then there exists a unique global strong solution to the Boltzmann equation [**1**] in the form [**3**] that satisfies

$$f(t,x,v) \in L^\infty_t L^2_v H^K_x \cap L^2_t N^{s,\gamma} H^K_x((0,\infty) \times T^3_x \times R^3_v).$$

With $\lambda > 0$, we have exponential decay to equilibrium:

$$\|f(t)\|_{L^2_v H^K_x} \le C e^{-\lambda t} \|f_0\|_{L^2_v H^K_x}.$$

We also have positivity; $F = \mu + \sqrt{\mu} f \ge 0$ if it is so initially.

The theorem above applies to the case when there is a spectral gap (28). In the weaker case of the very soft potentials $\gamma + 2s < 0$, our estimates constructively prove that there is no spectral gap. This resolves a conjecture in (28). To handle this second case in which there is no spectral gap, we will work in the Hilbert spaces $H^K_\ell(T^3_x \times R^3_v)$ where

$$\|f\|^2_{H^K_\ell} \stackrel{\text{def}}{=} \sum_{|\alpha|+|\beta| \le K} \int_{R^3_v} dv \int_{T^3_x} dx w^{\ell-|\beta|}(v) |\partial^\alpha_\beta f(x,v)|^2.$$

This is a standard Sobolev space. The weight corresponds to

$$w(v) = \langle v \rangle^{-\gamma-2s}.$$

Furthermore, we will consider the weighted nonisotropic space $N^{s,\gamma}_{\ell,K}$ with norm, $\|f\|^2_N$, given by

$$\sum_{|\alpha|+|\beta| \le K} \int_{T^3_x} dx \int_{R^3_v} dv \Big\{ \langle v \rangle^{\gamma+2s} w^{\ell-|\beta|}(v) |\partial^\alpha_\beta f(x,v)|^2 + \langle v \rangle^{\gamma+2s+1} w^{\ell-|\beta|}(v) \int_{R^3} dv' \frac{(\partial^\alpha_\beta f' - \partial^\alpha_\beta f)^2}{d(v,v')^{3+2s}} \mathbf{1}_{d(v,v') \le 1} \Big\}.$$

In both terms we are summing over $|\alpha| + |\beta| \le K$. Our multi-index notation for derivatives is $\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3} \partial^{\beta_1}_{v_1} \partial^{\beta_2}_{v_2} \partial^{\beta_3}_{v_3}$. Then we have the following theorem for the soft-potentials:

Theorem 3. Suppose $\gamma + 2s \in (-1, 0)$. Fix $K \ge 5$ and $\ell \ge 0$. Choose the initial state $f_0(x,v) \in H^K_\ell(T^3_x \times R^3_v)$ in [**3**] that satisfies [**4**]. There is an $\eta_0 > 0$ such that if $\|f_0\|_{H^K_\ell} \le \eta_0$ then there exists a unique global classical solution to the Boltzmann equation [1] in the form [3] that satisfies

$$f(t,x,v) \in L_t^\infty H_\ell^K \cap L_t^2 N_{\ell,K}^{s,\gamma}((0,\infty)\times T_x^3 \times R_v^3).$$

We have polynomial decay to any order $m \geq 0$:

$$\|f(t)\|_{H_\ell^K} \leq C_m(1+t)^{-m}\|f_0\|_{H_{\ell+m}^K}.$$

We also have positivity $F = \mu + \sqrt{\mu} f \geq 0$ if it is so initially.

In the context of the intermolecular potential [2], we give the full proof specifically in case of potentials $p > 3$ in ref. 26 and complete the rest of the cases, including $2 < p \leq 3$ in ref. 27. In the following we will explain our proof from ref. 26 because this already encapsulates the all of the essential mathematical difficulties associated with the full range of angular singularities in $b(\cos\theta)$.

Main Estimates. We will exposit our main estimates in the context $L^2(R_v^3)$ because this space already includes all of the singular geometric fractional derivative estimates on the velocity variables. To prove Theorem 2 the needed inequalities are of the form

$$|\langle \Gamma(g,h),f\rangle| \lesssim |g|_{L^2}|h|_{N^{s,\gamma}}|f|_{N^{s,\gamma}} + \text{l.o.t.} \quad [7]$$

$$\langle Lg,g\rangle \gtrsim |g|_{N^{s,\gamma}}^2 - \text{l.o.t.}, \quad [8]$$

where L^2 denotes the $L^2(R_v^3)$ norm. This proves to be a genuinely difficult task because the operators Γ and L are intimately connected and, among other consequences, this means that if both of these desired inequalities are simultaneously true then Hilbert space $N^{s,\gamma}$ satisfying these inequalities is unique. So the task is threefold: before proving each of the two fundamentally important inequalities the unique candidate for $N^{s,\gamma}$ must be identified. The precise version of the upper bound inequality [7] proved in ref. 26 is exactly

$$|\langle \Gamma(g,h),f\rangle| \lesssim |g|_{L^2}|h|_{N^{s,\gamma}}|f|_{N^{s,\gamma}} + |g|_{L^2_{\gamma+2s}}[|h|_{L^2}|f|_{N^{s,\gamma}} + |h|_{N^{s,\gamma}}|f|_{L^2}], \quad [9]$$

and the coercive inequality [8] is supplied by

$$\langle Ng,g\rangle \approx |g|_{N^{s,\gamma}}^2$$
$$|\langle Kg,g\rangle| \leq \eta|g|_{L^2_{\gamma+2s}}^2 + C_\eta|g|_{L^2}^2, \quad [10]$$

where $\eta > 0$ is any fixed, small number and $C_\eta > 0$.

Identification of the Space $N^{s,\gamma}$. As already mentioned, it turns out that the candidate Hilbert space $N^{s,\gamma}$ is a weighted, anisotropic Sobolev space. To be precise, we consider the metric on R^3 that is induced by the embedding $v \mapsto (v, \frac{1}{2}|v|^2) \in R^4$, i.e., we regard R^3 as a particular choice of parametrization of the paraboloid $(v, \frac{1}{2}|v|^2) \subset R^4$. If Δ_P is taken to be the metric Laplacian on this paraboloid, then the appropriate Hilbert space to consider is given by

$$|g|_{N^{s,\gamma}}^2 \approx \int_{R^3} dv \langle v\rangle^{\gamma+2s}|(I-\Delta_P)^{\frac{s}{2}} g|^2,$$

where the integral dv is just the usual Lebesgue measure on R^3. Rather than work directly with the operator $(I-\Delta_P)^{\frac{s}{2}}$, we find it more convenient to use a geometric Littlewood–Paley-type decomposition inspired by the work of Stein (29) and Klainerman–Rodnianski (30). We do not, however, take a semigroup approach to the actual construction of our Littlewood–Paley projections as Stein did. Instead, we use the embedding of the paraboloid in R^4 to our advantage. If $d\mu$ is the Radon measure on R^4 corresponding to surface measure on the paraboloid, our approach is to take a renormalized version of the *four-dimensional, Euclidean* Littlewood–Paley decomposition of the *measure $gd\mu$* as our nonisotropic, three-dimensional, Littlewood–Paley-type decomposition for the function g. Among other benefits, this approach automatically allows for a natural extension of the Littlewood–Paley projections P_jg and Q_jg as smooth functions defined on R^4 in a neighborhood of the paraboloid. This allows us to avoid a direct discussion of the induced metric on R^3 by phrasing our results in terms of the projections P_jg, Q_jg, and various Euclidean derivatives of these functions in R^4 instead of R^3. Another advantage is that we are able to obtain the appropriate Littlewood–Paley characterization of the space $N^{s,\gamma}$ by a pointwise comparison of the corresponding Littlewood–Paley square function to the integral expression for the norm.

The Upper Bound Inequality. The proof of the estimate [9] is based on a dyadic decomposition of the singularity of the collision kernel B as well as a Littlewood–Paley-type decomposition of the functions h and f. The end result is that one is led to consider a triple sum

$$|\langle \Gamma(g,h),f\rangle| \leq C \sum_{k=-\infty}^{\infty}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} |\langle \Gamma_k(g, Q_{j_1}h, Q_{j_2}f\rangle|.$$

Here, we have already performed an appropriate dyadic decomposition of the singularity in each Γ_k, each piece of the decomposition has the kernel $B_k(|v-v_*|, \cos\theta)$. Control over the sum of the pieces rests on two important observations:

1. When considering terms for which 2^{-k} is large relative to 2^{-j_1} and 2^{-j_2}, a favorable estimate holds simply because the support of $B_k(|v-v_*|,\cos\theta)$ is compact and bounded away from the singularity at $\theta = 0$. This is the regime that may be thought of as being far from the singularity.
2. When either 2^{-j_1} or 2^{-j_2} is large relative to 2^{-k}, i.e., near the singularity, an improvement may be made by exploiting the inherent cancellation structure of Γ_k. The cost that must be paid to use this cancellation is that derivatives must fall on either $Q_{j_1}h$ or $Q_{j_2}f$. In this case, it turns out that it is always possible to arrange for the derivatives to be placed on the function of our choice. Placing the derivatives on the function of largest scale (the function whose index is least) grants extra decay that allows one to sum all the terms by comparison to a geometric series.

This style of decomposition, of course, has a long history and generally follows the standard procedures of harmonic analysis. The essential unique feature in the above argument for the Boltzmann collision operator is that we do not measure cancellations in the usual isotropic way; cancellations are measured instead by using the metric on the paraboloid. It should be further noted that our analysis allows us to essentially ignore the dependence of $\Gamma(g,h)$ on the function g. This is a great advantage as it means that one may think of the trilinear form $\langle \Gamma(g,h),f\rangle$ as a family of bilinear forms in h and f parametrized by the function g. This observation is essential because the fully trilinear form falls well outside the scope of existing tools in harmonic analysis.

The Dual Formulation. A key point of significant technical importance in the proof of the upper bound inequality is that we must be able to make estimates for $\langle \Gamma(g,h),f\rangle$ that exploit the intrinsic cancellations at the cost of placing derivatives on any one of the two functions h or f that we choose. If we were not forced to consider fractional derivatives, a suitable tool would be integration-by-parts. As it stands however, it is necessary to find two different yet analogous representations of the trilinear form $\langle\Gamma(g,h),f\rangle$ that clearly relate cancellation to smoothness of h and f, respectively. It turns out that placing derivatives on f is fairly straightforward to do. In particular, one may apply a standard pre-post change of variables on the gain term $\mathcal{Q}^+$ to obtain the representation

$$\langle\Gamma(g,h),f\rangle = \int_{R^3} dv \int_{R^3} dv_* \int_{S^2} d\sigma B g_* h(M'_* f' - M_* f)$$

(that is justified by approximation of B by a sequence of cutoff kernels). Clearly, for each fixed g there is an operator T_g such that $\langle\Gamma(g,h),f\rangle = \langle T_g f,h\rangle$ and, moreover, the formula above can be used to write down an explicit formula for T_g. To place derivatives on h, on the other hand, it is necessary to derive a Carleman-type representation that involves only differences of h' and h, i.e., no differences of g or f. To that end, it is necessary to compute what we call the "dual formulation," because this amounts to writing down a formula for T_g^*. These computations may be found in the appendix of ref. 26; the end result is that

$$\langle\Gamma(g,h),f\rangle = \int_{R^3} dv \int_{R^3} dv_* \int_{S^2} d\sigma B g_* f' \times \left(M'_* h - M_* h' \frac{|v'-v_*|^3 \Phi(v'-v_*)}{|v-v_*|^3 \Phi(v-v_*)}\right).$$

An interesting consequence of this formula is that the gain term $\mathcal{Q}^+$ is unchanged and only the loss term $\mathcal{Q}^-$ differs in these two formulas. These two formulas also demonstrate the essentially straightforward dependence on g that we use to apply traditionally bilinear methods to the trilinear form.

The Coercive Inequality. The key to proving **[10]**, on the other hand, is to show that

$$\langle Ng,g\rangle \approx |g|^2_{N^{s,\gamma}}.$$

We prove an equivalent statement in terms of the Littlewood–Paley-type projections. This analysis consists of two parts. The first is a rewriting of the derivative part of the norm $\langle Ng,g\rangle$ by means of a Carleman-type representation as

$$\langle Ng,g\rangle = \int_{R^3} dv \int_{R^3} dv' K(v,v')(g'-g)^2 + \int_{R^3} dv \nu(v)|g(v)|^2,$$

for an appropriate function $K(v,v')$. As before, if we let $d(v,v')$ denote the Euclidean distance in R^4 between the points $(v,\frac{1}{2}|v|^2)$ and $(v',\frac{1}{2}|v'|^2)$, a simple pointwise estimation of this function K demonstrates that

$$K(v,v') \gtrsim (d(v,v'))^{-3-2s}(\langle v\rangle\langle v'\rangle)^{\frac{\gamma+2s+1}{2}},$$

for a large set of pairs (v,v'), the exact description of which is slightly complicated. The second part is to demonstrate that the set of pairs for which this inequality holds is large enough to conclude an integral version of this inequality, namely,

$$\langle Ng,g\rangle \gtrsim \int_{R^3} dv \int_{R^3} dv' (\langle v\rangle\langle v'\rangle)^{\frac{\gamma+2s+1}{2}} \frac{(g'-g)^2}{d(v,v')^{3+2s}} \mathbf{1}_{d(v,v')\le 1}.$$

This latter argument is accomplished by means of a partition of unity and Fourier analysis, the key point being that the expressions

$$\int_{R^3} dv \int_{R^3} dv' (g'-g)^2 \frac{\Omega(v-v')}{|v-v'|^{3+2s}},$$

are uniformly comparable for all $g \in H^s(R^3)$ as Ω ranges over the family of nonnegative, homogeneous functions of degree 0 for which $|\Omega|_{L^1(S^2)} \gtrsim 1$ and $|\Omega|_{L^\infty(S^2)} \lesssim 1$.

Local Existence. We establish the relevant local existence theorem by iteration taking $f^0(t,x,v) \overset{\text{def}}{=} f_0(x,v)$ and defining $f^{n+1}(t,x,v)$ for $n \ge 0$ in terms of the linear equation

$$(\partial_t + v\cdot\nabla_x + N)f^{n+1} + Kf^n = \Gamma(f^n, f^{n+1}),$$

with initial data $f^{n+1}(0,x,v) = f_0(x,v)$. This linear equation admits smooth solutions for suitable smooth small initial data. We define the "dissipation rate" as

$$\mathcal{D}(f(t)) \overset{\text{def}}{=} \sum_{|\alpha|\le K} \int_{T_x^3} dx |\partial_x^\alpha f(t,x)|^2_{N^{s,\gamma}},$$

and the total norm

$$\mathcal{G}(f(t)) \overset{\text{def}}{=} \|f(t)\|^2_{L_v^2 H_x^K} + \int_0^t d\tau \mathcal{D}(f(\tau)).$$

For appropriately small initial data $f_0 \in L_v^2 H_x^K$, these are shown to have uniformly bounded norm $\mathcal{G}(f^n(t))$ on some fixed time interval independent of n. Passing to the limit gives local existence:

Theorem 4. (Local existence). For any sufficiently small $M_0 > 0$, there exists a time $T^* = T^*(M_0) > 0$ and $M_1 > 0$, such that if

$$\|f_0\|^2_{L_v^2 H_x^K} \le M_1,$$

then there is a unique solution $f(t,x,v)$ to **[5]** on $[0,T^*)\times T_x^3 \times R_v^3$ such that

$$\sup_{0\le t\le T^*} \mathcal{G}(f(t)) \le M_0.$$

Furthermore, $\mathcal{G}(f(t))$ is continuous over $[0,T^*)$. Lastly, we have positivity if $F_0(x,v) = \mu + \mu^{1/2} f_0 \ge 0$ then $F(t,x,v) = \mu + \mu^{1/2} f(t,x,v) \ge 0$.

Global Existence. To establish global existence, we exploit the space-time macroscopic decomposition and nonlinear energy method due to Guo (14). By Boltzmann's H-theorem, the operator L is nonnegative and for every fixed (t,x) the null space of L is given by the five dimensional space

$$\mathcal{N} \overset{\text{def}}{=} \text{span}\{\sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, |v|^2\sqrt{\mu}\}.$$

Denote the orthogonal projection from $L^2(R_v^3)$ onto the null space $\mathcal{N}$ by $\mathbf{P}$. The energy estimate we obtain is as follows.

Theroem 5. Given the initial data $f_0 \in L^2(R_v^3 : H^K(T_x^3))$, for some $K \ge 3$, which satisfies **[4]** initially and the assumptions of Theorem 4. Consider the corresponding local solution, $f(t,x,v)$, to **[5]** that continues to satisfy **[4]**. There exists a small constant $M_0 > 0$ such that if

$$\|f(t)\|^2_{L_v^2 H_x^K} \le M_0 \qquad \textbf{[11]}$$

then there are universal constants $\delta > 0$ and $C_2 > 0$ such that

$$\mathcal{D}(\{\mathbf{I}-\mathbf{P}\}f(t)) \ge \delta\mathcal{D}(\mathbf{P}f(t)) - C_2\frac{d\mathcal{I}(t)}{dt},$$

where $\mathscr{I}(t)$ is the an appropriate "interaction functional" that is comparable to the energy.

This is really a statement of the linearized H-Theorem. We conclude in the usual way by defining

$$\mathscr{E}(t) \stackrel{\text{def}}{=} (C_1 + 1)\|f(t)\|^2_{L^2_v H^K_x} - C' \mathscr{I}(t),$$

for appropriate constants C_1 and C' (so that $\mathscr{E}(t)$ is equivalent to $\|f(t)\|^2_{L^2_v H^K_x}$) and showing that

$$\mathscr{E}(t) + \frac{\tilde{\delta}}{2}\int_0^t d\tau \mathscr{D}(\tau) \le \mathscr{E}(0)$$

for a suitable positive constant $\tilde{\delta}$, which proves that $\mathscr{E}(t)$ remains bounded for all time.

Entropy Production. We conclude with the observation that the nonisotropic nature of the linearized setting carries over to the fully nonlinear situation with the same geometry, at least in some broad cases. Precisely, the Boltzmann H-Theorem may be reexpressed as

$$\frac{dH}{dt}(t) = \int_{T^3} dx D(F) \ge 0,$$

where the nonnegative Entropy production functional is defined by

$$D(F) = -\int_{R^3} Q(F,F) \log F dv$$
$$= \frac{1}{4}\int_{R^3} dv \int_{R^3} dv_* \int_{S^2} d\sigma B(v - v_*, \sigma)(F'F'_* - FF_*) \log \frac{F'F'_*}{FF_*}.$$

This entropy production functional is zero if and only if it is operating on a Maxwellian equilibrium. This formally demonstrates that Boltzmann's equation defines an irreversible dynamics, and predicts convergence to Maxwellian in large time.

Coercive isotropic entropy production estimates in the non cutoff regime, such as (17), have found widespread utility. With our methods and nonisotropic Sobolev space, the following entropy production estimate may be established:

$$D(F) \gtrsim \int_{R^3} dv \int_{R^3} dv' \langle v \rangle^{\gamma+2s+1} \frac{(\sqrt{F'} - \sqrt{F})^2}{d(v,v')^{3+2s}} \mathbf{1}_{d(v,v') \le 1} - \text{l.o.t.}$$

This estimate follows straightforwardly from our estimates in ref. 26, Section 6 when combined with Eq. 29 from ref. 31 in the particular case where the unknown functions satisfy $F(v) \ge \varepsilon e^{-c|v|^2}$ for any given $\varepsilon, c > 0$. Remarkably, this is the same semi-norm as in the linearized context, and it is a stronger nonisotropic and nonlocal semi-norm than the one found in the local smoothing estimate from ref. 17, stronger, that is, in terms of the weight power multiplied on the order of differentiation.

Final Remark. After the completion of this present work, the authors of ref. 32 sent us a preprint that proves a related global existence result for similar perturbations of the Maxwellian equilibrium states [**3**] in the whole space. Their result currently applies to the case of Maxwell molecules and moderate angular singularities. For the inverse power intermolecular potentials, this corresponds to the case $p = 5$. Their methods of proof are substantially different from our own.

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