and from (2.12) and (3.1) in (3.2), we find on equating coefficients of y^l that the functions Q^i_{jkl} are the components of a tensor which, being defined in projective coördinates, is independent of the choice of the affine connection.

Differentiation of (2.2) and evaluation at $y^i = 0$ by means of (2.7) and (2.8) give

$$Q^i_{jkl} = E^i_{jkl} - \frac{1}{3} S\left(E^i_{jkl} - 2\frac{\delta^i_l E^\alpha_{jk\alpha}}{n-1}\right) + \frac{\delta^i_j E^\alpha_{kl\alpha} + \delta^i_k E^\alpha_{jl\alpha}}{n-1},$$

where

$$E^i_{jkl} = \frac{\partial \Pi^i_{jk}}{\partial x^l} + \Pi^\alpha_{jk}\,\Pi^i_{\alpha l},$$

and S ( ) indicates the sum of the terms obtained by permuting j, k, l cyclically. The Weyl projective curvature tensor[3] is given by

$$W^i_{jkl} = Q^i_{jkl} - Q^i_{jlk}.$$

We leave the question of the other projective invariants given by (3.1) to be treated in a subsequent paper.

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[1] Cf. Veblen, these PROCEEDINGS, vol. 8, p. 192.

[2] These quantities were used by T. Y. Thomas in his paper on equi-projective geometry of paths in this volume of these PROCEEDINGS. The parameter which we introduce by means of equations (2.4) is essentially his projective parameter. We had found it before seeing his paper.

[3] Cf. Weyl, *Göttingen Nach.*, 1921, p. 99; also J. M. Thomas, this volume of these PROCEEDINGS.

NOTE ON THE PROJECTIVE GEOMETRY OF PATHS

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As remarked by T. Y. Thomas in this number of these PROCEEDINGS, the quantities

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{\delta^i_j}{n+1}\Gamma^\alpha_{\alpha k} - \frac{\delta^i_k}{n+1}\Gamma^\alpha_{\alpha j} \qquad (1)$$

have a projective significance in that their value is the same for all affine connections associated with a given system of paths. It is the purpose of this note to show that the projective curvature tensor discovered by Weyl (*Göttingen Nachrichten*, 1921, p. 99) can be expressed in terms of them, and also that they can be used to construct new tensors.

When the coördinates x are subjected to an arbitrary analytic transformation, we have

$$\frac{\partial \bar{x}^i}{\partial x^\alpha}\Pi^\alpha_{jk} = \frac{\partial \bar{x}^\alpha}{\partial x^j}\frac{\partial \bar{x}^\beta}{\partial x^k}\bar{\Pi}^i_{\alpha\beta} + \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} + \frac{1}{n+1}\frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial \log \Delta}{\partial x^k} + \frac{1}{n+1}\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \log \Delta}{\partial x^j}, \quad (2)$$

where Δ is the jacobian of the transformation. Differentiation of this relation with respect to x^l followed by an interchange of k and l and a subtraction gives equations which are equivalent to

$$\mathfrak{B}^i_{jkl} = \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k}\frac{\partial \bar{x}^\delta}{\partial x^l}\bar{\mathfrak{B}}^\alpha_{\beta\gamma\delta} + \delta^i_l\, b_{jk} - \delta^i_k\, b_{jl}, \quad (3)$$

where

$$\mathfrak{B}^i_{jkl} = \frac{\partial \Pi^i_{jk}}{\partial x^l} - \frac{\partial \pi^i_{jl}}{\partial x^k} + \Pi^\alpha_{jk}\,\Pi^i_{\alpha l} - \Pi^\alpha_{jl}\,\Pi^i_{\alpha k}, \quad (4)$$

and

$$b_{jk} = \frac{1}{n+1}\left(\Pi^\alpha_{jk}\frac{\partial \log \Delta}{\partial x^\alpha} - \frac{\partial^2 \log \Delta}{\partial x^j \partial x^k} - \frac{1}{n+1}\frac{\partial \log \Delta}{\partial x^j}\frac{\partial \log \Delta}{\partial x^k}\right).$$

Contracting for i and l in (3) we get

$$b_{jk} = \frac{1}{n-1}\left(\mathfrak{B}^\alpha_{jk\alpha} - \frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k}\bar{\mathfrak{B}}^\alpha_{\beta\gamma\alpha}\right).$$

This result, substituted in (3), gives

$$W^i_{jkl} = \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k}\frac{\partial \bar{x}^\delta}{\partial x^l}\bar{W}^\alpha_{\beta\gamma\delta}, \quad (5)$$

where

$$W^i_{jkl} = \mathfrak{B}^i_{jkl} + \frac{\delta^i_k}{n-1}\mathfrak{B}^\alpha_{jl\alpha} - \frac{\delta^i_l}{n-1}\mathfrak{B}^\alpha_{jk\alpha}. \quad (6)$$

Equations (5) show that the functions defined by (6) are the components of a tensor which is projective in character since it is defined in terms of the Π's. Substitution from (1) and (4) in (6) gives an expression easily identified with that for the Weyl projective curvature tensor. (Cf. Veblen and Thomas, *Trans. Amer. Math. Soc.*, vol. 25, p. 560.)

Thus the conditions of integrability of the equations (2) of transformation of the Π's give rise to the projective curvature tensor in much the same way that the equations of transformation of the Γ's give rise to the ordinary curvature tensor.

If we differentiate equations (5) with respect to x^p and eliminate the second derivatives of the $\bar{x}$'s by means of (2), we find it is not possible to eliminate the derivatives of Δ by means of the process of contraction alone. It is readily shown, however, that the following quantities are the components of a projective tensor:

$$W^j_{abc}W^k_{fgh}W^l_{qst}W^p_{uvw}\left(\mathfrak{B}^i_{jklp} - \frac{\delta^i_p}{n-2}\mathfrak{B}^\alpha_{jkl\alpha}\right)$$
$$+ \frac{1}{n-2}(2W^i_{jkl}W^j_{abc}W^k_{fgh}W^l_{qst}\mathfrak{B}^\alpha_{uvw\alpha} + W^i_{pkl}W^k_{fgh}W^l_{qst}W^p_{uvw}\mathfrak{B}^\alpha_{abc\alpha}$$
$$+ W^i_{jpl}W^j_{abc}W^l_{qst}W^p_{uvw}\mathfrak{B}^\alpha_{fgh\alpha} + W^i_{jkp}W^k_{abc}W^k_{fgh}W^p_{uvw}\mathfrak{B}^\alpha_{qst\alpha}),$$

where $\mathfrak{B}^i_{jklp}$ is formed from $\mathfrak{B}^i_{jkl}$ in the same way as its covariant derivative but with Π's replacing the Γ's. That the tensor does not vanish identically has been verified by computing one of its components in a special case.

Although this tensor is of a highly complicated form, it is of interest since it indicates the existence of projective tensors other than the Weyl curvature tensor.

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DETERMINATION OF THE STRESSES IN A BEAM BY MEANS OF THE PRINCIPLE OF LEAST WORK[1]

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Introduction.—We owe to Saint-Venant the development of the theory of torsion and bending of beams. In two classic memoirs,[2] equally remarkable for their completeness and lucidity, he has given a rigorous solution of this problem, comprising in particular the determination of displacements, strains and stresses in cylindrical or prismatic beams fixed at one end and loaded at the other.

When the displacements are given, it is an easy matter, by means of the general equations of elastic equilibrium and by a simple differentiation to determine the forces which produce them.

The inverse problem, to determine the displacements when the forces are given, is of far more practical importance, but had not, when Saint-Venant took up the problem, been solved in a general way, on account of the difficulty of integrating the equations in which the displacements enter and of determining the functions and arbitrary constants of integration.