SPECIAL FEATURE: INTRODUCTION

Cluster algebras

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What Is a Cluster Algebra?

Cluster algebras were conceived by Fomin and Zelevinsky (1) in the spring of 2000 as a tool for studying dual canonical bases and total positivity in semisimple Lie groups. However, the theory of cluster algebras has since taken on a life of its own, as connections and applications have been discovered in diverse areas of mathematics, including representation theory of quivers and finite dimensional algebras, cf., for example, refs. 2–15; Poisson geometry (16–19); Teichmüller theory (20–24); string theory (25–31); discrete dynamical systems and integrability (6, 32–38); and combinatorics (39–47).

Quite remarkably, cluster algebras provide a unifying algebraic and combinatorial framework for a wide variety of phenomena in these and other settings. We refer the reader to the survey papers (36, 48–53) and to the cluster algebras portal (www.math.lsa.umich.edu/~fomin/cluster.html) for various introductions to cluster algebras and their links with other subjects in mathematics (and physics).

In brief, a cluster algebra $\mathcal{A}$ of rank k is a subring of an ambient field $\mathcal{F}$ of rational functions in k variables, say $x_1, \ldots, x_k$. Unlike most commutative rings, a cluster algebra is not presented at the outset via a complete set of generators and relations. Instead, from the data of the initial seed—which includes the k initial cluster variables $x_1, \ldots, x_k$, plus an exchange matrix—one uses an iterative procedure called "mutation" to produce the rest of the cluster variables. In particular, each new cluster variable is a rational expression in $x_1, \ldots, x_k$. The cluster algebra is then defined to be the subring of $\mathcal{F}$ generated by all cluster variables.

The set of all cluster variables has a remarkable combinatorial structure: It is a union of overlapping algebraically independent k subsets of $\mathcal{F}$ called "clusters," which together have the structure of a simplicial complex called the "cluster complex." The clusters are related to each other by birational transformations of the following kind: For every cluster $\mathbf{x}$ and every cluster variable $x \in \mathbf{x}$, there is another cluster $\mathbf{x}' = (\mathbf{x} - \{x\}) \cup \{x'\}$, with the new cluster variable x' determined by an exchange relation of the form

$$xx' = y^+M^+ + y^-M^-.$$

Here y^+ and y^- are coefficients, whereas M^+ and M^- are monomials in the elements of $\mathbf{x} - \{x\}$.

Example: The Type *A* Cluster Algebra

Although we have not given the formal definition of cluster algebra, we will nevertheless provide an example, which we hope will give the reader the flavor of the theory. The combinatorics of triangulations of an n-gon (a convex polygon with n vertices) will be used to describe the example presented here. We will subsequently identify the resulting cluster algebra with the homogeneous coordinate ring of the Grassmannian $Gr_{2,n}$ of 2-planes in an n-dimensional vector space.

Fig. 1 shows an example of a triangulation T, with $n = 8$. We have labeled the diagonals of T by the numbers 1, 2, . . ., 5 and the sides of the octagon by the numbers 6, 7, . . ., 13. We now set $\mathcal{F} = \mathbb{Q}[x_6, \ldots, x_{13}](x_1, \ldots, x_5)$. In other words, $\mathcal{F}$ consists of rational functions in the variables $x_1, \ldots, x_5$ labeled by the diagonals, with coefficients which are polynomials in the variables $x_6, \ldots, x_{13}$ labeled by the sides. The variables $x_1, \ldots, x_5$ are the initial cluster variables. The variables $x_6, \ldots, x_{13}$ generate the ring of coefficients, and we regard $\mathcal{F}$ as an algebra over $\mathbb{Q}[x_6, \ldots, x_{13}]$.

We then use the combinatorics of triangulations and flips of triangulations to define the other cluster variables. Consider a triangulation T containing a diagonal t. Within T, the diagonal t is the diagonal of some quadrilateral. Then there is a new triangulation T' which is obtained by replacing t with the other diagonal of that quadrilateral. This local move is called a "flip."

Consider the graph whose vertex set is the set of triangulations of an n-gon, with an edge between two vertices whenever the corresponding triangulations are related by a flip. It is well-known that this flip graph is connected, and moreover is the 1-skeleton of a convex polytope called the "associahedron." See Fig. 2 for a picture of the flip graph of the hexagon.

Now we can associate a cluster variable (an element of $\mathcal{F}$) to each of the diagonals of the n-gon by imposing a relation for every flip: Given a quadrilateral, with sides a, b, c, and d and diagonals e and f, we stipulate that $x_ex_f = x_ax_c + x_bx_d$ (Fig. 3). Using the fact that the flip graph is connected (i.e., we can get from the initial triangulation to any other by a series of flips), it is clear that we can attach in this way an element x_i of $\mathcal{F}$ to each diagonal i. It is an exercise to show that this construction is well defined, that is, x_i does not depend on the sequence of flips used to pass from the initial triangulation to any triangulation containing the diagonal i. By definition, the cluster algebra $\mathcal{A}_n$ associated with the n-gon is the subalgebra of $\mathcal{F}$ generated by all of the cluster variables associated with its diagonals. A cluster of $\mathcal{A}_n$ is a subset of the set of cluster variables corresponding to the diagonals of a triangulation of the n-gone. Thus, the rank of $\mathcal{A}_n$ is $n - 3$. Note that our construction depends on a choice of initial triangulation. However, if we choose two different triangulations, the resulting cluster algebras will be isomorphic.

Let $Gr_{2,n}$ be the Grassmann variety parametrizing 2-planes in an n-dimensional complex vector space. We now explain how the cluster algebra $\mathcal{A}_n$ is related to the homogeneous coordinate ring $\mathbb{C}[Gr_{2,n}]$ of $Gr_{2,n}$ in its Plücker embedding.

Recall that $\mathbb{C}[Gr_{2,n}]$ is generated by Plücker coordinates p_{ij} for $1 \leq i < j \leq n$. The relations among the Plücker coordinates are generated by three-term Plücker relations: For any $1 \leq i < j < k < \ell \leq n$, one has

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk}. \quad [1]$$

To make the connection with cluster algebras, label the vertices of an n-gon from 1 to n in order around the boundary. Then each side and diagonal of the polygon is uniquely identified by the labels of its endpoints. This gives a bijection between the set of Plücker coordinates and the set of sides and diagonals of the n-gon (Fig. 1, *Right*). By

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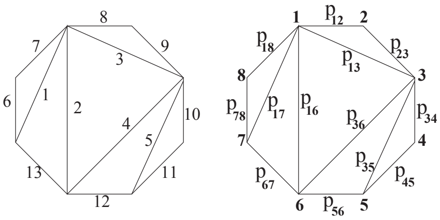

Fig. 1. A triangulation *T* of an octagon (*Left*), and the labeling of *T* by Plücker coordinates (*Right*).

noting that the Plücker relations correspond to exchange relations in $\mathcal{A}_n$ (Fig. 4), one shows that there is a well-defined isomorphism from $\mathcal{A}_n \otimes \mathbb{C}$ to $\mathbb{C}[Gr_{2,n}]$ mapping the elements x_i associated with the sides and diagonals to the corresponding Plücker coordinates.

One may generalize this example in several ways. First, one may replace $Gr_{2,n}$ by an arbitrary Grassmannian, or partial flag variety. It turns out that the homogeneous coordinate ring $\mathbb{C}[Gr_{k,n}]$ of any Grassmannian has the structure of a cluster algebra (54), and more generally, so does the multihomogeneous coordinate ring of any partial flag variety $SL_m(\mathbb{C})/P$ (55). Second, one may generalize this example by replacing the *n*-gon (topologically a disk with *n*-marked points on the boundary) by an orientable Riemann surface *S* (with or without boundary) together with some marked points *M* on *S*. One may still consider triangulations of (*S*, *M*), and use the combinatorics of these triangulations to define a cluster algebra. This cluster algebra is closely related to the decorated Teichmüller space associated to (*S*, *M*) (56, 57).

Our example of the cluster algebra $\mathcal{A}_n$ may be misleading for the following reason: It has finitely many cluster variables. In general, a cluster algebra may have infinitely many cluster variables. Those that have only finitely many cluster variables are said to be of finite type. There is a beautiful classification of finite-type cluster algebras (58): It turns out that their classification is parallel to the celebrated Cartan–Killing classification of semisimple Lie algebras. In other words, the finite-type cluster algebras are classified by Dynkin diagrams. In this classification, the cluster algebra $\mathcal{A}_n$ has type A_{n-3}. Note that the cluster structure of $\mathbb{C}[Gr_{k,n}]$ is generally of infinite type when $k > 2$.

Cluster Algebras at the Mathematical Sciences Research Institute

Over the dozen years that passed since its inception, the theory of cluster algebras attracted into its realm many excellent researchers from all around the world. The semester-long Mathematical Sciences Research Institute (MSRI) program in Berkeley on Cluster Algebras held during August 20–December 21, 2012 featured extended stays of more than 50 mathematicians, ranging from graduate students and postdoctoral students to senior researchers. Many additional mathematicians passed through the cluster algebras program for shorter visits, for example, to participate in one of the three focused workshops that took place as part of the program.

The program presented a broad panorama of the current state of this rapidly expanding subject, enabled many mathematicians to broaden their understanding of the roles that cluster algebras play in various active areas of research, facilitated new interactions, and, more generally, provided the participants an excellent opportunity to share and further develop their ideas. This MSRI program was the last long-term mathematical activity for our dear friend and mentor Andrei Zelevinsky, who passed away in April 2013. We dedicate this Special Feature to his memory.

We shall now proceed to describe the content of the papers that appear in this issue of PNAS. Our goal is to illustrate the flavor and breadth of the research being done in the field of cluster algebras. We refer the reader to the papers themselves for full details.

Webs on Surfaces, Rings of Invariants, and Clusters

Let $V \cong \mathbb{C}^k$ be a vector space endowed with a volume form. The topic of the first paper in our issue, by Fomin and Pylyavskyy (59), is rings of SL_k invariants of collections of vectors, covectors, and matrices, in a *k*-dimensional vector space. More precisely, let $R_{a,b,c}(V) = \mathbb{C}[(V^*)^a \times V^b \times (\mathrm{SL}(V))^c]^{\mathrm{SL}(V)}$ be the ring of $\mathrm{SL}(V)$-invariant polynomials on $(V^*)^a \times V^b \times (\mathrm{SL}(V))^c$. A generating set for this ring has been known since 1976, but the size of a minimal generating set, let alone an efficient description of the corresponding ideal of relations, are, so far, unknown.

In ref. 59, the authors study the case $k = 3$, and give a series of remarkable results and conjectures about the structure of $R_{a,b,c}(V)$. First, they construct marked surfaces *S* of type (*a*, *b*, *c*), which they then connect to the ring of invariants $R_{a,b,c}(V)$. More precisely, they define tensor diagrams, which are certain bipartite graphs embedded in *S*, and define the skein algebra of tensor diagrams in *S*. They then give a surjective ring homomorphism from the skein algebra to $R_{a,b,c}$,

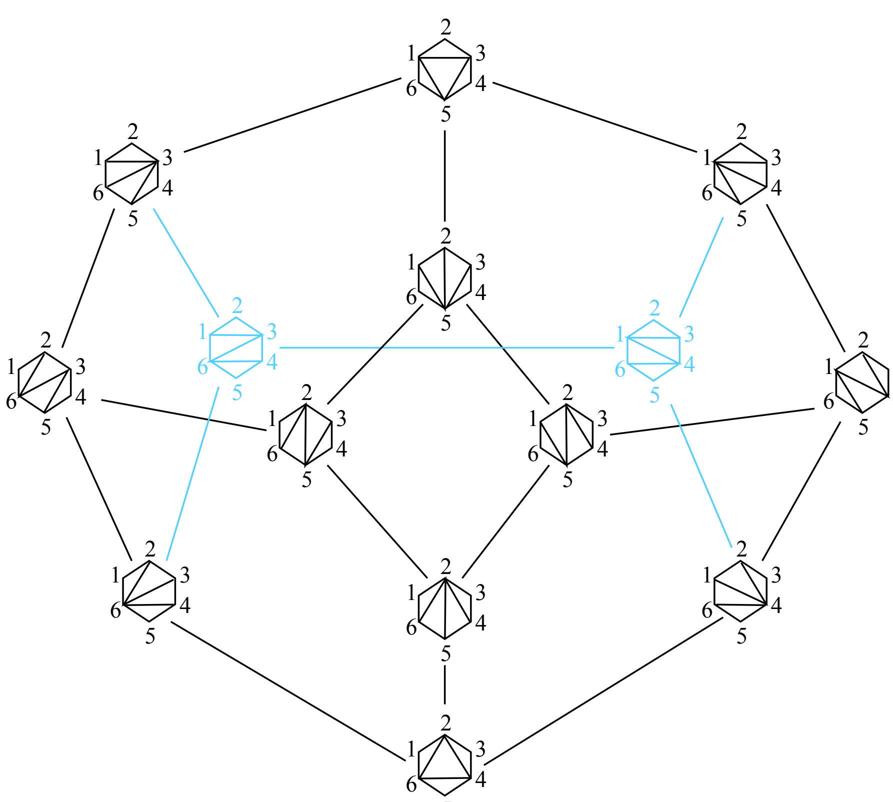

Fig. 2. The exchange graph of the cluster algebra associated with the hexagon, which coincides with the 1-skeleton of the associahedron.

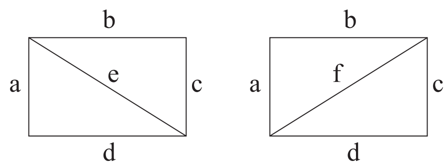

Fig. 3. A flip in a quadrilateral. The corresponding exchange relation is $x_e x_f = x_a x_c + x_b x_d$.

which they conjecture is injective. The authors use the combinatorics of tensor diagrams to construct a cluster algebra which lies inside of $R_{a,b,c}$. In general these cluster algebras are of infinite mutation type and hence should be completely intractable. Remarkably, the authors suggest a conjectural combinatorial description of all of the cluster variables.

The Cremmer–Gervais Cluster Structure on SL_n

One of the prototypical examples of cluster algebras described by Berenstein et al. (60) is the coordinate ring of a (double Bruhat cell of a) simple complex Lie group. This example is directly related to the original motivations for cluster algebras coming from total positivity and canonical bases. The same example was also studied by Gekhtman et al. (19) from the point of view of Poisson geometry. Poisson–Lie structures on a complex Lie group have been classified by Belavin and Drinfeld (61). It turns out that the usual cluster structure is compatible with the so-called standard Poisson–Lie structure. The purpose of the second paper in this issue of PNAS, by Gekhtman et al. (62), is to present, for the Lie group SL_n, a very different cluster structure compatible with a nonstandard Poisson–Lie structure due to Cremmer–Gervais. One remarkable feature of this cluster structure is that, in contrast to the standard one, the cluster algebra is strictly contained in its upper bound. Another one is that the totally positive part of SL_n with respect to this exotic cluster structure is strictly contained in the usual set of totally positive matrices.

Quantum Cluster Algebras and Quantum Nilpotent Algebras

The existence of cluster structures on coordinate rings of Poisson–Lie groups makes it natural to investigate the possibility of quantizing the notion of a cluster algebra. A general axiomatic definition of a quantum cluster algebra was given by Berenstein and Zelevinsky (63), together with a conjectural quantum cluster structure on the coordinate ring of an arbitrary double Bruhat cell. In the third paper (64), Goodearl and Yakimov announce a proof of this conjecture, and present the main features of their construction. They work in the general framework of a quantum nilpotent algebra, a large class of noncommutative rings endowed with a torus action. Quantum nilpotent algebras are unique factorization domains, and Goodearl and Yakimov show the existence of a canonical quantum cluster structure whose initial cluster consists of an appropriate sequence of prime elements. They then explain how their main theorem yields quantum cluster algebra structures on quantum Schubert cells and quantum double Bruhat cells.

Introduction to τ-Tilting Theory

After the pioneering paper of Marsh et al. (2) first pointing out relationships between cluster algebras and the representation theory of quivers, many authors investigated these deep connections. Marsh et al. found striking similarities between cluster mutation and the classical notion of tilting in the representation theory of algebras. The fourth paper in this issue of PNAS (65), by Iyama and Reiten, starts with a beautiful survey of tilting theory, then proceeds with cluster-tilting theory, a very successful variant motivated by cluster algebras. This is meant as an introduction to a new notion called "τ-tilting." Here τ stands for the Auslander–Reiten translation of a finite-dimensional algebra Λ. A Λ-module is called τ-tilting if it is τ-rigid (a concept introduced long ago by Auslander and Smalo, ref. 66) and is maximal for this property. Surprisingly, τ-tilting modules give rise to a new operation of mutation occurring in any finite-dimensional algebra Λ. In the final section, the authors relate τ-tilting with cluster tilting when Λ is a cluster-tilted algebra (or more generally a 2-Calabi-Yau tilted algebra).

Greedy Bases in Rank 2 Quantum Cluster Algebras

A famous conjecture of Fomin and Zelevinsky (1) [recently proved by Cerulli Irelli et al. (12) for a large class of cluster algebras called "skew symmetric"] states that the cluster monomials form a linear independent subset of a cluster algebra. There has been great interest in the problem of finding natural bases of the cluster algebra containing this subset and satisfying some strong positivity properties. For rank 2 cluster algebras, Lee et al. (67) have described such a basis, which consists of so-called greedy elements with a beautiful combinatorial description. The fifth paper, by Lee et al. (68), reviews this construction, then establishes the existence of a quantum lift of the greedy basis in any rank 2 quantum cluster algebra. However, the quantum greedy elements are not always universally positive. The paper closes with a series of exciting open problems.

Cluster-Like Coordinates in Supersymmetric Quantum Field Theory

One of the recent unexpected appearances of cluster algebras is in quantum field theory, a fast-moving branch of theoretical physics. In our sixth paper, Neitzke (69) gives a review of one way in which quantum field theory and cluster algebras interact. More specifically, ref. 69 concerns $\mathcal{N}=2$ supersymmetric quantum field theories in four dimensions. These theories have associated hyperkähler moduli spaces, and these moduli spaces carry a structure which looks like an extension of the notion of cluster variety. In particular, one encounters the usual variables and mutations from the usual cluster algebra story, along with more exotic extra variables and generalized mutations. Perhaps the most exciting aspect of ref. 69 is that it argues that geometric ideas coming from quantum field theory lead to a natural extension of the theory of cluster algebras.

A Positive Basis for Surface Skein Algebras

The Jones polynomial of a knot is one of the simplest and most important knot invariants at the center of many recent advances in topology; it is a polynomial in one variable q. The skein algebra of a surface is a natural generalization of the Jones polynomial to knots that live in a thickened surface. The seventh paper in this issue of PNAS, by Thurston (70), discusses the positivity properties of three different bases of the skein algebra (at $q=1$): the bangles basis, the band basis, and the bracelet basis. Here a basis $\{x_i\}$ of an algebra A over $\mathbb{Z}$ is called "positive" if $x_i x_j = \sum_k m_{ij}^k x_k$, with $m_{ij}^k \geq 0$ for any i, j. It is known that the bangles basis is not a positive basis. Fock and Goncharov conjectured that the bracelet basis is positive in their groundbreaking work (section 12 in ref. 20). The main result of ref. 70 is a proof of Fock–Goncharov's conjecture. Because the bracelet basis contains the cluster monomials for the cluster algebra associated to the surface, this result is closely related to the corresponding instance of the strong positivity conjecture for cluster algebras.

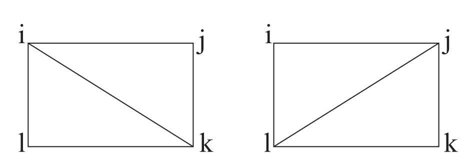

Fig. 4. A flip in a quadrilateral and the corresponding exchange relation $p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk}$.

Additionally, ref. 70 conjectures that the band basis is a positive basis precisely when the surface $\sum$ has either nonempty boundary or at least one puncture, i.e., if the fundamental group $\pi_1(\sum)$ is free. Several intriguing open problems are also proposed.

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