A new invariant of 4-manifolds

Robion Kirby[a,1,2] **and Abigail Thompson**[b,1]

[a]Department of Mathematics, University of California, Berkeley, CA 94720; and [b]Department of Mathematics, University of California, Davis, CA 95616

Edited by David Gabai, Princeton University, Princeton, NJ, and approved August 14, 2018 (received for review October 30, 2017)

We define an integer invariant L_X of a smooth, compact, closed 4-manifold X by minimizing a certain complexity of a trisection of X over all trisections. The good feature of L_X is that when $L_X = 0$ and X is a homology 4-sphere, then X is diffeomorphic to the 4-sphere. Naturally, L is hard to compute.

trisections | 4-manifolds | Heegaard splittings | curve complex

We define an integer invariant L_X of a smooth, compact, closed 4-manifold X by minimizing a certain complexity of a trisection of X over all trisections.

Loops in the Cut Complex

Let X be a closed, orientable, smooth 4-manifold. In ref. 1, Gay and Kirby show that X has a trisection into three 4-dimensional handlebodies and prove that any two trisections of X are stably equivalent under a suitable notion of stabilization. We exploit these results to define a new 4-manifold invariant L_X and prove that $L_X = 0$ if and only if X is a connect sum of copies of $S^1 \times S^3$, $S^2 \times S^2$, CP^2, and S^4 (the case of the empty connect sum). If $L_X \leq 1$, we obtain the same 4-manifolds, so L_X is never one.

Definition 1: A$(g; k_1, k_2, k_3)$-trisection of a closed, oriented 4-manifold X (where $0 \leq k_i \leq g, i = 1, 2, 3$) is a decomposition $X = X_1 \cup X_2 \cup X_3$, where (*i*) each $X_i \cong \natural^{k_i} S^1 \times B^3$, (*ii*) each $X_i \cap X_j \cong \natural^g S^1 \times B^2$ (for $i \neq j$), and (*iii*) $X_1 \cap X_2 \cap X_3 \cong \#^g S^1 \times S^1$.

Definition 2: A $(g; k_1, k_2, k_3)$-trisection diagram is a 4-tuple $(\Sigma, \alpha, \beta, \gamma)$ such that each of (Σ, α, β), (Σ, β, γ), and (Σ, γ, α) are genus g Heegaard diagrams of $\#^k{}_i S^1 \times S^2, i = 1, 2, 3$, respectively. A trisection diagram for a given trisection $X = X_1 \cap X_2 \cap X_3$ is a trisection diagram $(\Sigma, \alpha, \beta, \gamma)$, where Σ is diffeomorphic to $X_1 \cap X_2 \cap X_3$, α is a cut system for $X_1 \cap X_2$, β for $X_2 \cap X_3$, and γ for $X_3 \cap X_1$.

The stabilization operation for a balanced trisection increases the genus of the central surface Σ by 3. It can be understood in terms of the trisection diagram by taking the connect sum of $(\Sigma, \alpha, \beta, \gamma)$ with the standard genus three trisection diagram of S^4.

An unbalanced trisection can be "balanced" by taking the connect sum with genus one trisections of the 4-sphere.

The topology of each of the three pieces of X is completely determined by a single integer k_i, and the topology of each of the overlaps between pieces is determined by another integer g. If $k = k_1 = k_2 = k_3$, the trisection is called balanced.

Given a trisection of X^4, we have a central surface $\Sigma = X_0 \cap X_1 \cap X_2$ in X bounding three 3-dimensional handlebodies $X_i \cap X_j$, which fit together in pairs to form Heegaard splittings of three 3-manifolds in X, and these 3-manifolds in turn uniquely bound three 4-dimensional 1-handlebodies. We can thus specify a trisection by considering systems of curves on Σ.

Definition 3: A cut system for a closed surface Σ of genus g is an unordered collection of g simple closed curves on Σ that cut Σ open into a $2g$-punctured sphere.

Definition 4: A genus g Heegaard diagram for a closed orientable 3-manifold is a triple (Σ, α, β), where Σ is a closed orientable genus g surface and each of α and β is a cut system for Σ.

Following Wajnryb (2) and Johnson (3), we define the following:

Definition 5: The cut complex $\mathcal{C}$ of Σ_g is a 1-complex with vertices corresponding to (isotopy classes) of cut systems. Two vertices α and α' in $\mathcal{C}$ are connected by an edge of type 0 if their corresponding cut systems $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_g\}$ and $\alpha' = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_g\}$ agree on $g - 1$ curves and their final curves are disjoint. Two vertices α and α' are connected by an edge of type 1 if their corresponding cut systems α and α' agree on $g - 1$ curves and their final curves intersect in a single point. The distance between two vertices α and β, $d(\alpha, \beta)$, is the length of the shortest path (using the edge-metric) connecting them in the cut complex.

Notice that if α and α' are connected by a type 0 edge, then α can be obtained from α' by sliding α_g over some of $\alpha_1, \alpha_2, \ldots, \alpha_{g-1}$. $\mathcal{C}$ is connected (4).

Suppose we are given a $(g; k_1, k_2, k_3)$-trisection diagram $(\Sigma, \alpha, \beta, \gamma)$ for a trisection $\mathcal{T}$ of X.

Definition 6: Let Γ_α be the set of all vertices in $\mathcal{C}$ that are path connected to α by type 0 edges (generalized handle slides). Define Γ_β and Γ_γ similarly (see Fig. 1).

Definition 7: We say two cut systems α and β are in good position with respect to each other if we can order each, $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_g, \beta = \beta_1, \beta_2, \ldots, \beta_g$, so that for each i, either α_i is parallel to β_i (and we write $\alpha_i \mathcal{P} \beta_i$) or α_i intersects β_i in exactly one point (and we write $\alpha_i \mathcal{D} \beta_i$), and α_i is disjoint from β_j for all $i \neq j$. We say α_i and β_j are a good pair if they are either parallel or intersect in a single point and are disjoint from all other αs and βs.

Note that it is possible for α, β, γ to pairwise all be in a good position but not with respect to the same ordering. For example, in Fig. 2 all pairs are in a good position, but α_1 is paired with γ_2 and α_2 with γ_1.

Every vertex in Γ_α represents a different cut system describing the same handlebody $X_1 \cap X_2$.

We can calculate the length of the shortest path between Γ_α and Γ_β. We use a mild generalization of Waldhausen's theorem for Heegaard splittings of the 3-sphere (5):

Theorem 8: Let (Σ, α, β) be a genus g Heegaard diagram for $\#^k S^1 \times S^2$. Then there exist cut systems α' and β' that are connected to α and β, respectively, through type 0 edges such that α' and β' are in good position with respect to each other.

Significance

All known 4-manifolds invariants cannot distinguish a possible counterexample to the smooth 4-dimensional Poincare Conjecture from the standard 4-sphere. The L invariant, defined in this paper, can do so, for if it vanishes on a homotopy 4-sphere X, then X must be diffeomorphic to the 4-sphere. Unfortunately, it is very hard to calculate.

Author contributions: R.K. and A.T. performed research and wrote the paper.

The authors declare no conflict of interest.

This article is a PNAS Direct Submission.

[1]R.K. and A.T. contributed equally to this work.

[2]To whom correspondence should be addressed. Email: kirby@math.berkeley.edu.

Published online October 22, 2018.

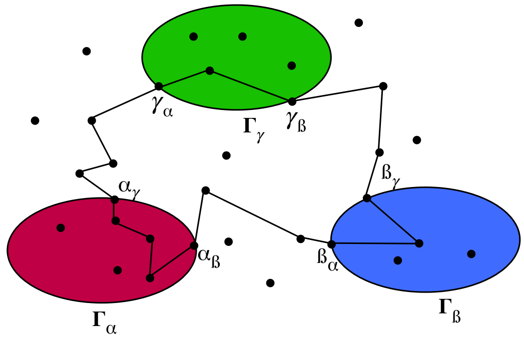

Fig. 1. $S^2 \times S^2$.

The details of this theorem appear in ref. 6, p. 313, together with a discussion of its relation to the isotopy question. We include an outline of the argument, which proceeds by induction, for the convenience of the reader. Proof: Use Haken's (7) lemma to find an essential separating 2-sphere S that intersects Σ in a single essential simple closed curve λ bounding imbedded disks E_α and E_β on both sides of Σ. Use an outermost arc on, say, E_α of intersections with the disks bounded by the αs to dictate handle slides on the αs to reduce the number of points of intersection between λ and the αs. This implies that any Heegaard splitting of $\#^k S^1 \times S^2$ is diffeomorphic to a good Heegaard splitting (but not necessarily isotopic to one).

Hence there exists a cut system in Γ_α that has distance precisely $g - k_1$, through $g - k_1$ type 1 edges, to the nearest cut system in Γ_β. Note that stabilizing the trisection increases the length of such a path in a straightforward way; if the initial trisection is balanced and the stabilization is balanced, the length of the path goes up by 2. For an unbalanced stabilization, the length goes up by either 0 or 1.

Definition 9: Let $l_{X,\mathcal{T}}$ be the length of the shortest closed path in $\mathcal{C}$ that intersects each of Γ_α, Γ_β, and Γ_γ, which also satisfies the following:

i) There are three pairs—$(\alpha_\beta, \beta_\alpha)$, $(\beta_\gamma, \gamma_\beta)$, and $(\gamma_\alpha, \alpha_\gamma)$—in

$$(\Gamma_\alpha, \Gamma_\beta), (\Gamma_\beta, \Gamma_\gamma), (\Gamma_\gamma, \Gamma_\alpha),$$

respectively, which are all good, so it takes $g - k_i$ type 1 moves to travel from the vertex corresponding to one element in the pair to the other.

ii) The subpath of $l_{X,\mathcal{T}}$ connecting α_β to α_γ (respectively, β_α to β_γ, γ_β to γ_α) remains within Γ_α (respectively, Γ_β, Γ_γ).

Normalize l by defining:

Definition 10: $L_{X,\mathcal{T}} = l_{X,\mathcal{T}} - 3g + k_1 + k_2 + k_3$. Note that this number can only decrease when we stabilize. Note also that this number is equal to the total number of type 0 moves in each of Γ_α, Γ_β, and Γ_γ.

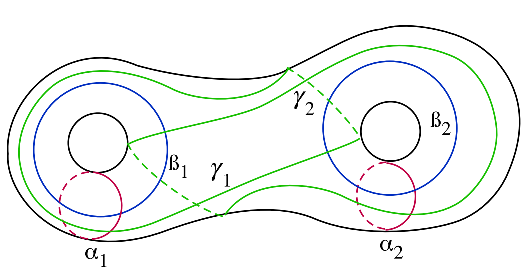

Fig. 2. $\mathcal{C}$.

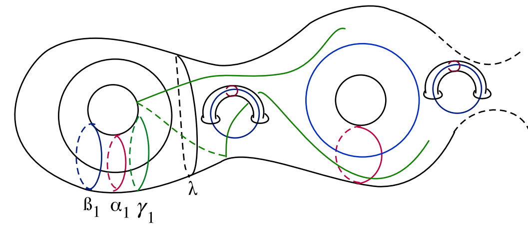

Fig. 3. $\alpha_1 \mathcal{P} \beta_1 \mathcal{P} \gamma_1$.

Definition 11: The length of X, denoted L_X, is the minimum value of $L_{X,\mathcal{T}}$ over all trisections $\mathcal{T}$ of X.

It follows immediately from the stable equivalence of balanced trisections (1) that L_X is well-defined.

We analyze the manifolds for which $L_X = 0$:

Theorem 12: $L_X = 0$ if and only if X is diffeomorphic to a connect-sum of copies of $S^1 \times S^3$, $S^2 \times S^2$, CP^2, and S^4 (in the case of an empty connect sum).

As an immediate corollary, we have the following:

Corollary 13: If X is a homology 4-sphere, then $L_X = 0$ if X is diffeomorphic S^4.

Proof of theorem: Let $(\Sigma, \alpha, \beta, \gamma)$ be a (g, k_1, k_2, k_3) trisection of X that realizes $L_{X,\mathcal{T}} = 0$.

Then, $g - k_1 = d(\alpha, \beta)$, $g - k_2 = d(\gamma, \beta)$, and $g - k_3 = d(\alpha, \gamma)$. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $(\beta_1, \beta_2, \ldots, \beta_n)$, $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, and $n = 1, \ldots, g$ be the curves corresponding to the cut systems α, β, γ.

Since $(\Sigma, \alpha, \beta, \gamma)$ realizes $L_{X,\mathcal{T}} = 0$, we may assume that $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\beta_1, \beta_2, \ldots, \beta_g$ are in good position with respect to each other and that $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\gamma_1, \gamma_2, \ldots, \gamma_g$ are in good position with respect to each other (any ordering of α determines one for β and for γ). Note that the βs and γs would also be "good" if we allowed reordering of subindices. Consider the example of $S^2 \times S^2$ where β_1 and γ_2 are good, as are β_2 and γ_1.

We may also assume that $\alpha_i \mathcal{P} \beta_i$ for $i = 1, \ldots, k_1$.

After possible relabeling, we have the following cases:

Case 1 : $\alpha_1, \beta_1, \gamma_1$ are all parallel (see Fig. 3).

No other curve from $\alpha \cup \beta \cup \gamma$ intersects $\alpha_1, \beta_1, \gamma_1$. Let δ be a simple closed curve intersecting α_1 (also β_1, γ_1) transversely in a single point, chosen to be disjoint from all other αs and βs. α_1 and δ together have a neighborhood that is a punctured torus T. We say that T is defined by α_1 and δ. Let $\lambda = \partial T$. λ is disjoint from all αs and βs but may intersect $\gamma_2 \cup \ldots \cup \gamma_g$. However, we can slide these γs over γ_1 to remove these intersections, obtaining $\gamma_2', \ldots, \gamma_g'$, which are disjoint from ∂T. Let $\gamma' = \gamma_1, \gamma_2', \ldots, \gamma_g'$. Since this operation has no effect on the intersections of curves with subindices $2, \ldots, g$, $(\Sigma, \alpha, \beta, \gamma')$ also realizes $L = 0$. In $(\Sigma, \alpha, \beta, \gamma')$, ∂T is a splitting curve—that is, a separating simple closed curve—disjoint from all curves in all three cut systems, which splits the diagram into two subdiagrams, each with $L = 0$. The subdiagram containing α_1 yields an $S^1 \times S^3$ summand that we can

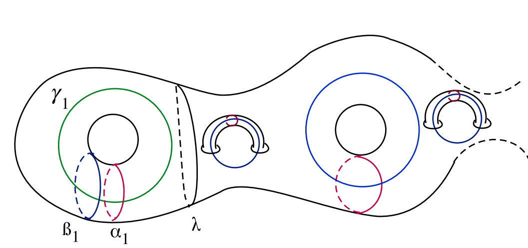

Fig. 4. $\alpha_1 \mathcal{P} \beta_1 \mathcal{D} \gamma_1$.

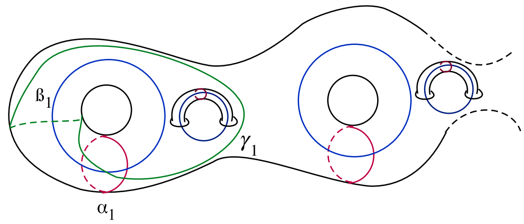

Fig. 5. $\alpha_1 \mathcal{D} \beta_1 \mathcal{D} \gamma_1 \mathcal{D} \alpha_1$.

split off and proceed to consider the smaller genus remaining subdiagram.

Case 2 : $\alpha_1 \mathcal{P} \beta_1$ and γ_1 intersects each in exactly one point (see Fig. 4).

As before, we can find a punctured torus T containing $\alpha_1, \beta_1, \gamma_1$, which in this case is automatically disjoint from all curves in $\alpha \cup \beta \cup \gamma$. Hence, there is an obvious S^4 summand in the trisection diagram, which we split off to reduce the genus and again proceed on the remainder.

Case 3 : No pair of curves from α, β, γ is parallel. In particular, $\alpha_1 \mathcal{D} \beta_1$ and $\alpha_1 \mathcal{D} \gamma_1$. Let λ be the boundary of the torus T defined by α_1 and β_1.

Subcase a: $\gamma_1 \mathcal{D} \beta_1$ (see Fig. 5).

Claim: Then we can split off a $\pm CP^2$ summand.

Proof: If γ_1 does not lie in T, then we can move it there by a type 0 move. Then, ∂T will be a splitting curve.

Subcase b: γ_1 is disjoint from β_1 (see Fig. 6). Then, we can assume (by relabeling as needed) that $\gamma_1 \mathcal{D} \beta_2$ and γ_1 are disjoint from all other curves in α and β.

Claim: Then, we can split off a $S^2 \times S^2$ summand.

Proof: We analyze the remainder of the γ_is and show there must exist a γ_2 such that

- $\gamma_2 \mathcal{D} \beta_1$,
- $\gamma_2 \cap \alpha_1$ is empty,
- $\gamma_2 \mathcal{D} \alpha_2$, and
- $\gamma_2 \cap \beta_2$ is empty.

This follows because exactly one γ, which we label γ_2, is dual to β_1, and it links γ_1 in λ when both intersect λ. That forces γ_2 to intersect α_2 in one point. By type 0 moves on γ_1 and γ_2, we can arrange that all curves with indices 1 or 2 are outside the punctured $S^2 \times S^2$, whose boundary is a splitting curve. This concludes the proof.

We now prove the stronger theorem:

Theorem 14: If there exists a trisection $\mathcal{T}$ such that $L_{X,\mathcal{T}} = 1$, then $L_X = 0$, and X is again diffeomorphic to a connect sum of copies of $S^1 \times S^3$, $S^2 \times S^2$, and CP^2.

Assume $(\Sigma, \alpha, \beta, \gamma)$ realizes $L_{X,\mathcal{T}} = 1$. We may also assume that $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\beta_1, \beta_2, \ldots, \beta_g$ are in good position with respect to each other and that $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\gamma_1, \gamma_2, \ldots, \gamma_g$ are in good position with respect to each other. Note that the βs and γs would also be good if we allowed reordering of subindices, with the exception of a single γ_j. Hence there exists a cut system $\Gamma' = \gamma_1, \gamma_2, .., \gamma_j', .., \gamma_g$ that is distance one from $\gamma_1, \gamma_2, \ldots, \gamma_g$ and that is good with respect to $\beta_1, \beta_2, \ldots, \beta_g$ after reordering.

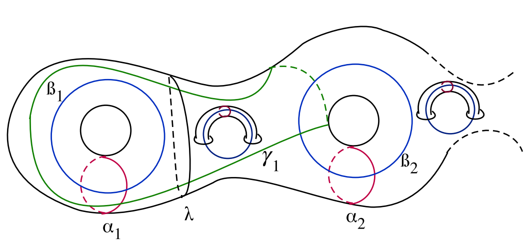

Fig. 6. $\alpha_1 \mathcal{D} \beta_1$, $\gamma_1 \mathcal{D} \alpha_1$.

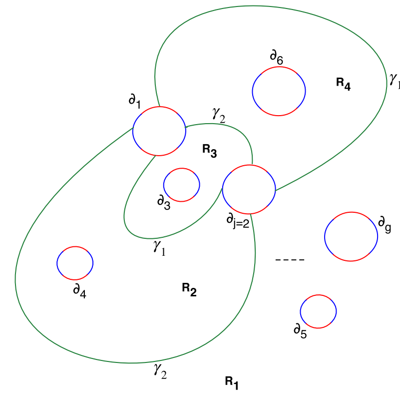

Fig. 7. *P*.

The arguments in cases 1 and 2 of theorem 1 work as before if $\alpha_i \mathcal{P} \beta_i$ or $\alpha_i \mathcal{P} \gamma_i$ for any i, or $\beta_i \mathcal{P} \gamma_k$ for any $k \neq j$, or $\beta_i \mathcal{P} \gamma_j'$; that is, we can assume that the trisection is balanced and that $g = k$ so that each X_i is a 4-ball.

If $g = 2$, the theorem follows from ref. 8. Assume $g > 2$. We relabel so $\Gamma = \gamma_1, \gamma_2, .., \gamma_g$ and $\Gamma' = \gamma_1, \gamma_2, .., \gamma_g'$.

Hence, we have the following string of relations:

$$\beta_1 \mathcal{D} \alpha_1 \mathcal{D} \gamma_1 \mathcal{D} \beta_j \mathcal{D} \alpha_j.$$

If $j = 1$, we are back in case 3, subcase a of the previous argument. Assume $j \neq 1$. Then, we can continue our string to the left,

$$\gamma_a \mathcal{D} \beta_1,$$

and to the right,

$$\alpha_j \mathcal{D} \gamma_b.$$

If $a = b$, then $a \neq g$. If $a \neq b$, then either a or b (or both) is not equal to g. In any case, we obtain a slightly longer string by adding on to the left or to the right, say to the left,

$$\gamma_a \mathcal{D} \beta_1 \mathcal{D} \alpha_1 \mathcal{D} \gamma_1 \mathcal{D} \beta_j \mathcal{D} \alpha_j,$$

where $a \neq g$ and γ_g is not in the string.

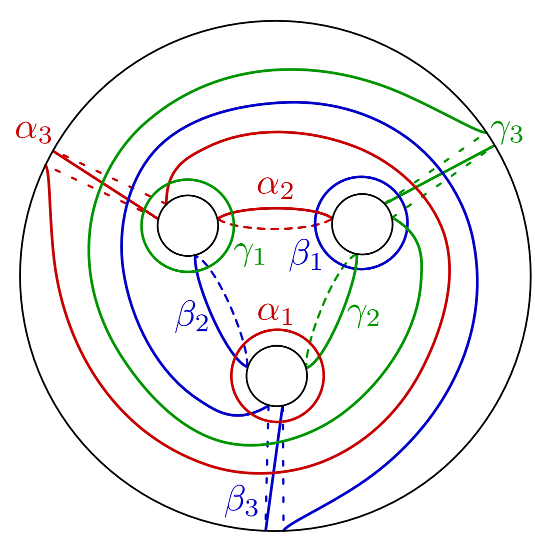

Fig. 8. *Q*.

Claim: $a = j$.

Proof of claim: Cut Σ open along all αs and βs to obtain a planar surface P with g boundary components, $\partial_1, \ldots, \partial_g$, with the labelling inherited from the αs. The remnants of γ_1 in P are two properly imbedded arcs connecting ∂_1 to ∂_j. The remnants of γ_a in P are also two properly imbedded arcs, whose endpoints are linked on ∂_1 with the endpoints of γ_1. We know γ_a is dual to exactly one α and disjoint from all others; the only available α that yields a single connected curve is α_j. Hence, $a = j$ (see Fig. 7).

We now relabel so $j = 2$ and summarize our findings thus far:

$$\gamma_2 \mathcal{D} \beta_1 \mathcal{D} \alpha_1 \mathcal{D} \gamma_1 \mathcal{D} \beta_2 \mathcal{D} \alpha_2 \mathcal{D} \gamma_2.$$

Definition 15: Call such a set $\gamma_2, \beta_1, \alpha_1, \gamma_1, \beta_2, \alpha_2, \gamma_2$ a good sextet.

The remnants of γ_1 and γ_2 in P cut P into four regions, one of which, R_1, contains ∂_g.

Suppose the other three regions, R_2, R_3, R_4, are disks—that is, contain no other boundary components of P.

Claim: Either γ_g or γ_g' is disjoint from $R_2 \cup R_3 \cup R_4$.

Proof of claim: Suppose γ_g intersects R_2. γ_g is disjoint from all αs except α_g and disjoint from all other γs, so γ_g can only intersect the pieces of ∂R_2 corresponding to remnants of the βs. Hence, (possibly after removing trivial intersections) γ_g intersects $\mathbb{R}_2$ in a collection of parallel arcs connecting the two β remnants on ∂R_2. This means that γ_g must also intersect R_3 and R_4 in a similar fashion. Recall that γ_g' is disjoint from γ_g, and γ_g' is disjoint from all βs except β_g. Then, by the same argument, if γ_g' intersects $\mathbb{R}_2$ at all, it must do so in a collection of parallel arcs connecting the two α remnants on ∂R_2. But any such arc would intersect an arc of γ_g, and γ_g' is disjoint from γ_g. So if γ_g intersects R_i, $i = 2, 3, 4$, then γ_g' cannot and vice versa.

Assume γ_g is disjoint from $R_2 \cup R_3 \cup R_4$.

Then, ∂R_1 is a splitting curve for α, β, γ, and we proceed by examining the smaller diagram inside R_1.

Suppose one of R_2, R_3, R_4 is not a disk, say R_2.

Using previous arguments, we can find another good sextet inside R_2.

This sextet also divides P into four components, one of which contains ∂_g.

If all other components are disks, we are done by the previous argument. Otherwise, select one that is not a disk, and repeat.

Eventually, we find a sextet such that one component of P defined by the sextet contains ∂_g, and all others are disks.

An Example with $L \leq 6$

Currently, the smallest nonzero $L_{X,\mathcal{T}}$ we know, namely 6, is achieved by a smooth orientable 4-manifold Q that is the quotient $(S^2 \times S^2)/Z/2$, where the group $Z/2$ acts by sending (x, y) to $(-x, -y)$. This allows the possibility that our theorem holds for $L \leq 5$, but we only conjecture the theorem can be strengthened to show that $L \leq 2$ implies $L = 0$. There is a notable lack of low-genus simply-connected, closed, smooth 4-manifolds (other than those with $L = 0$). In the nonspin case, there are connected sums of $\pm CP^2$, and in the spin case, there is the $K3$ complex surface. Many of these manifolds have exotic smooth structures (e.g., CP^2 with at least two points blown up; this means connected summing with $-CP^2$s), but these have complicated handlebody structures suggesting L is large. For π_1 nonzero, our Q is a fairly simple example and is a natural candidate for the smallest nonzero L.

There is a handlebody description of Q obtained by taking a simple description of the nonorientable disk bundle over RP^2 and doubling it to get Q (see ref. 9, p. 27). There are algorithms to turn this handlebody description into a trisection with genus three, but the diagram in Fig. 8 will most easily show that $L \leq 6$. This diagram was discovered independently by David Gay and by Jeff Meier, in the latter case as part of studying trisections of twist spun 3-manifolds.

By symmetry, it suffices to calculate how many type 0 moves are required to make the αs and βs standard. Ignore the γ curves, and observe that α_2 and β_1 are a pair that intersect each other once and are disjoint from all other αs and βs. Notice next that α_1 and β_2 would be a good pair if not for the fact that β_3 intersects α_1 twice. These intersections can be removed by two handle slides of β_3 over β_2. First, push the closer point of intersection clockwise along α_1 and then slide over β_2 to remove the point of intersection. Then, do the same with the further point of intersection, again moving clockwise and sliding over β_2.

We now have two pairs intersecting once each, and then one can check that α_3 and β_3 are in fact parallel on Σ, and thus, the $\alpha - \beta$ curves form a standard Heegaard spitting of $S^1 \times S^2$, as desired.

A sharp reader might observe that if the second handle slide had been done counterclockwise, then the two handle slides would combine into one type 0 move, suggesting that $l = 3$, but a sharper reader will realize that in this case α_3 and β_3 are no longer parallel, for the other pairs are stuck between the otherwise parallel curves.

Remarks

It seems likely that a complicated handlebody diagram for X would lead to a large value of L. But it is sobering to realize that the complex hypersurfaces such as the $K3$ surface are not connected sums of smaller 4-manifolds, yet if one connect sums with one copy of CP^2, the resulting complicated handlebody slides away to a connected sum of $\pm CP^2$s (refs. 10 or 11), showing that there must also be a way to do handle slides on the αs, βs, and γs to get $L = 0$.

ACKNOWLEDGMENTS. Both authors were supported in part by the National Science Foundation.

1. Gay DT, Kirby R (2016) Trisecting 4-manifolds. *Geom Topol* 20:3097–3132.
2. Wajnryb B (1998) Mapping class group of a handlebody. *Fund Math* 158:195–228.
3. Johnson J (2006) Heegaard splittings and the pants complex. *Algebr Geom Topol* 6:853–874.
4. Hatcher A, Thurston W (1980) A presentation for the mapping class group of a closed orientable surface. *Topology* 19:221–237.
5. Waldhausen F (1968) Heegaard-Zerlegungen der 3-Sphare. *Topology* 7:195–203.
6. Schleimer S (2007) *Waldhausen's theorem. Geometry & Topology Monographs*, eds Gordon C, Moriah Y (Mathematical Sciences Publishers, Berkeley, CA), Vol 12, pp 299–317.
7. Haken W (1968) Some results on surfaces in 3-manifolds. *Studies in Modern Topology*, ed Hilton PJ (Mathematical Association of America, Prentice-Hall, Englewood Cliffs, NJ), pp 39–98.
8. Meier J, Zupan A (2017) Genus-two trisections are standard. *Geom Topol* 21:1583–1630.
9. Kirby RC (1989) *The Topology of 4-Manifolds*. Springer Lecture Notes (Springer, Heidelberg), Vol 1374.
10. Mandelbaum R, Moishezon B (1976) On the topological structure of non-singular algebraic surfaces in CP^3, *Topology* 15:23–40.
11. Harer J, Kas A, Kirby R (1986) *Handlebody Decompositions of Complex Surfaces*. Memoirs of the American Mathematical Society (American Mathematical Society, Providence, RI), Vol 62, Number 350.