to any angle with vertex at the origin, there is a line J in the given angle, or in it f(z) converges uniformly to zero; (II) if $\{a_n\}$ clusters to a given ray, the ray is a line J, or on it f(z) converges to zero.

[1] Presented to American Mathematical Society, October 31, 1931.

[2] NATIONAL RESEARCH FELLOW.

[3] NATIONAL RESEARCH FELLOW.

[4] G. Julia, *Leçons sur les fonctions uniformes à point singulier essentiel isolé*, Paris, chap. V, 102–118 (1924).

[5] P. Montel, *Leçons sur les familles normales des fonctions analytiques*, Paris (1927).

[6] See L. Bieberbach, *Math. Zeit.*, **3**, 175–190 (1919).

[7] See, however, G. Valiron, *Acta Mathematica*, **52**, 67–92 (1928), and H. Milloux, *Ibid.*, **52**, 189–255 (1929).

[8] See F. Iversen, *Compt. Rend. (Paris)*, **166**, 156 (1918).

[9] See E. Lindelöf, *Acta. Soc. Fenn.*, **35**, Nr. 7 (1908).

[10] See S. Mandelbrojt, *J. Math. (Liouville)*, **18**, 176 (1929).

[11] See Gontcharoff, *Compt. Rend. (Paris)*, **185**, 1100–1102 (1927).

ON THE TIME AVERAGE THEOREM IN DYNAMICS

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1.—*Introductory.*—We consider a conservative dynamical system with an analytic Hamiltonian function $H(p_1, \ldots p_n, q_1, \ldots q_n)$. The phases $p_1, \ldots p_n, q_1, \ldots q_n$ of the system may be regarded as points P in the phase space. After elapse of the time t a point $P = P_0$ will be carried into a new position $P_t = T_t(P)$. Apart from singularities, $T_t(P)$ is well known to represent an analytic one parameter group of analytic one to one transformations of the phase space into itself,

$$T_sT_t = T_{s+t}, (P_t)_s = P_{t+s}.$$

The phase volume is invariant under T_t. As H = const. is an integral of the equations of motion, T_t implies a group on the $(2n - 1)$-dimensional manifold $H = c$. Similarly, we obtain a group on the $(2n - m - 1)$-dimensional manifold $H = c, H_v = c_v$, if the equations $H_v = c_v, v = 1_1 \ldots, m_1$ represent m further known integrals. In any case, a positive integral $m = \int \rho d\sigma, \rho > 0$, invariant under T_t, is associated with the considered manifold. We suppose that the group is free of singularities, i.e., that T_t is analytically determined for all values of t. This situation occurs in well-known cases, for instance, in the case of the regularized problem of three bodies.

Generally, let Ω be an analytic manifold, and let $P \longrightarrow P_t$ $(P_s)_t = P_{s+t}$ be an analytic group of analytic one to one transformations of Ω into itself. A measure m in the sense of Lebesgue is supposed to exist on Ω and to be invariant under those transformations. The well-known notation

$$(f, g) = \int_\Omega f(P)\overline{g(P)}dm, \quad ||f|| = \sqrt{(f, f)}$$

is used. A function $f(P)$ with finite $||f||$ represents a point in the Hilbert space, and the transition from the function $f(P)$ to the function $f(P_t)$ represents a linear and unitary transformation $U_t(f)$ of the Hilbert space into itself,

$$U_t(af + bg) = aU_t(f) + bU_t(g); \quad (U_tf, U_tg) = (f, g), \qquad (1)$$

the unitarity being due to the invariance of the measure m. The $U_t(f)$ form a linear one parameter group of unitary transformations,

$$U_tU_s = U_{s+t}. \qquad (2)$$

This connection between the motions of a dynamical system and linear groups of unitary transformations in the Hilbert space has been recently pointed out by B. O. Koopman.[1]

In this paper the author gives a simple and elementary proof of a remarkable theorem recently proved by J. v. Neumann in connection with the above correspondence principle.[2] In the dynamical case this theorem concerns the existence of the "time average" $f^*(P)$ of a function $f(P)$,

$$f^*(P) = \lim_{T=\infty} i.\ m.\ \frac{1}{T}\int_0^T f(P_t)\,dt,$$

in the sense of convergence in the mean on Ω. v. Neumann's general theorem runs as follows:

THEOREM. *For any given point f of the Hilbert space an "average point" f^* is associated with the set of points $U_t(f)$, $-\infty < t < +\infty$, in such a way that*

$$\lim_{T=\infty}\left\|\frac{1}{T}\int_\alpha^{\alpha+T} U_tfdt - f^*\right\| = 0.$$

If, in the dynamical case, the function $f(P)$ is the characteristic function of a measurable point set A, $f = 1$ on A, $f = 0$ on $\Omega - A$, f^* represents the mean time in which the wandering point P_t crosses A. Professor Birkhoff discovered quite recently the remarkable fact that, in the case where A is an open point set, f^* is not only the limit in the mean, but the actual limit apart from a point set of zero measure.[3] The existence of this mean time of crossing establishes the actual proof of a fundamental hypothesis of classical statistical mechanics.

v. Neumann's proof of the above theorem is based on the spectral representation of the operators U_t,

$$(U_t f, g) = \int_{-\infty}^{+\infty} e^{i\lambda t}\, d(E_\lambda f, g),$$

due to M. Stone. The spectral form is of some interest in connection with dynamics, since it opens a way toward the Fourier analysis of the general motion. It throws also a sidelight on convergence questions in dynamics and celestial mechanics. In general, the spectrum of a dynamical system will be a continuous one, i.e., the function $\varphi(\lambda) = (E_\lambda f, g)$ will not be a pure step function. This shows that in general the motions of dynamical systems cannot be approximated by trigonometric sums in any sense of uniformity within $-\infty < t < +\infty$. However, as far as concerns the proof of v. Neumann's above theorem, the usage of the spectral theory is avoidable.

2. *Proof of the Theorem.*—We start with two preliminary remarks due to v. Neumann (loc. cit.) For any point f of the Hilbert space the integral $\int_a^b U_t f dt$ is a well-defined point of that space. Furthermore, we have

$$\left(\int_a^b U_t f dt,\ g\right) = \int_a^b (U_t f, g)\, dt.$$

We set

$$\varphi_{T,a} = \frac{1}{T}\int_a^{a+T} U_t f dt.$$

According to the Riesz-Fischer theorem we have to prove that

$$\|\varphi_{T',a'} - \varphi_{T,a}\| \longrightarrow 0,\ T, T' \longrightarrow \infty.$$

We have

$$\|\varphi_{S,b} - \varphi_{T,a}\|^2 = \begin{matrix} (\varphi_{S,b}, \varphi_{S,b}) + (\varphi_{T,a}, \varphi_{T,a}) \\ - (\varphi_{S,b}, \varphi_{T,a}) - \overline{(\varphi_{S,b}, \varphi_{T,a})}, \end{matrix} \tag{3}$$

where the expressions on the right-hand side may be evaluated as follows.

$$\|\varphi_{S,b} - \varphi_{T,a}\|^2 = \frac{1}{ST}\left(\int_b^{b+S} U_s f ds,\ \int_a^{a+T} U_t f dt\right) = \frac{1}{ST}\int_b^{b+S}\int_a^{a+T} (U_s f, U_t f)\, dt ds.$$

By setting

$$g(\tau) = (U_t f, f), \; F(\tau) = \frac{1}{\tau}\int_0^\tau g(t)dt, \; G(\tau) = \frac{1}{2}\left(F(\tau) + \overline{F(\tau)}\right) \quad (4)$$

and observing that $(U_s f, U_t f) = (U_{s-t} f, f) = g(s - t)$, we readily obtain

$$(\varphi_{S,b}, \varphi_{T,a}) = \frac{1}{ST}\int_{S-T-\delta}^{S-\delta} \tau F(\tau)d\tau - \frac{1}{ST}\int_{-T-\delta}^{-\delta} \tau F(\tau)d\tau; \; \delta = a - b.$$

The second term on the right may be transformed by taking into account the equations

$$g(\tau) = (f, U_{-\tau}f) = \overline{(U_{-\tau}f, f)} = \overline{g(-\tau)}, f(\tau) = \overline{f(-\tau)}.$$

We obtain

$$(\varphi_{S,b}, \varphi_{T,a}) = \frac{1}{ST}\int_{S-T-\delta}^{S-\delta} \tau F d\tau + \frac{1}{ST}\int_{\delta}^{\delta+T} \tau \overline{F} d\tau; \; \delta = a - b.$$

The application of this equation to the four terms on the right-hand side in (3) leads to our final formula

$$\frac{1}{2}\left\| \varphi_{S,b} - \varphi_{T,a} \right\|^2 = \left[\begin{array}{c} \frac{1}{S^2}\int_0^S + \frac{1}{T^2}\int_0^T \\ - \frac{1}{ST}\int_{S-T-\delta}^{S-\delta} - \frac{1}{ST}\int_{\delta}^{T+\delta} \end{array} \right] G(\tau)\tau d\tau. \quad (5)$$

We now proceed as follows. According to the Schwarz inequality $g(\tau)$ is a bounded function, $|g| \leqq ||f||\cdot||U_\tau f|| = ||f||^2 = \gamma$,

$$|g(\tau)| \leqq \gamma, \; |G(\tau)| \leqq \gamma. \quad (6)$$

We have

$$(S - \delta - s)G(S - \delta - s) = SG(S) - \int_{S-\delta-s}^{S} g(\tau)\, d\tau$$
$$\geqq SG(S) - \gamma(|s| + |\delta|),$$

thus yielding

$$\int_{S-T-\delta}^{S-\delta} G\tau d\tau = \int_0^T (S - \delta - s)G(S - \delta - s)ds \quad (7)$$
$$\geqq STG(S) - \frac{1}{2}\gamma\,(T + |\delta|)^2.$$

Furthermore we have

$$\left| \int_{\delta}^{T+\delta} G\tau\, d\tau \right| \leqq \frac{1}{2}\gamma\,(T + |\delta|)^2. \quad (8)$$

If we set

$$G^* = \lim_{\tau = \infty} \sup G(\tau)$$

the inequality $G(\tau) < G^* + \frac{1}{8}\epsilon^2$, ϵ being a given positive number, holds for all sufficiently large values of τ. Hence we conclude that

$$\frac{1}{\tau^2}\int_0^\tau G(\tau)\tau d\tau < \frac{1}{2}\left(G^* + \frac{\epsilon^2}{8}\right), \tau > N(\epsilon). \tag{9}$$

From (5), (7), (8) and (9)

$$\frac{1}{2}\,||\varphi_{S,b} - \varphi_{T,a}||^2 \leqq G^* - G(S) + \frac{\epsilon^2}{8} + \gamma\,\frac{(T + |\delta|)^2}{ST} \tag{10}$$

if $S, T < N(\epsilon)$. According to the inequality $\frac{1}{2}||f_1 + f_2||^2 \leqq ||f_1||^2 + ||f_2||^2$ we have

$$||\varphi_{T',a'} - \varphi_{T,a}||^2 \leqq 2\,||\varphi_{S,b} - \varphi_{T,a}||^2 + 2\,||\varphi_{S,b} - \varphi_{T',a'}||^2$$

and according to (10), $\delta = a - b$, $\delta' = a' - b$,

$$||\varphi_{T',a'} - \varphi_{T,a}||^2 \leqq \epsilon^2 + 8(G^* - G(S)) + \frac{4}{S}\,\gamma\left\{\frac{(T + |\delta|)^2}{T} + \frac{(T' + |\delta'|)^2}{T'}\right\}$$

for $S, T, T' > N(\epsilon)$. If $S \longrightarrow \infty$ in such a way that $G(S) \longrightarrow G^*$, we finally obtain

$$||\varphi_{T',a'} - \varphi_{T,a}|| \leqq \epsilon, \text{ if } T, T' > N(\epsilon),$$

which completes the proof.

From (2) we infer the following property of the linear operators $\varphi_{T,a}$,

$$U_\tau\,\varphi_{T,0} = \varphi_{T,\tau}.$$

This shows that $f^* = U_\tau f^*$ holds for any τ, i.e., that f^* is invariant under the group U_t. For an arbitrary invariant point h of the Hilbert space we obtain the equations

$$(\varphi_{T,0}(f), h) = \frac{1}{T}\int_0^T (U_t f, h) = \frac{1}{T}\int_0^T (f, U_{-t}h)dt = (f, h),$$

thus yielding

$$(f^*, h) = (f, h). \tag{11}$$

(11) holds for any invariant h and determines f^* uniquely.

3. *Remarks on the Quasiergodic Dynamical Case.*—In the special case of a dynamical system equation (11) has been used by v. Neumann (loc. cit.) for the determination of the time average $f^*(P)$ of a function $f(P)$ by means of a "space average." The quasiergodic case, which plays a fundamental rôle in classical statistical mechanics, is characterized by the condition that either $m(A) = 0$ or $m(A) = m(\Omega)$ holds for any measurable and invariant subset A of Ω, $m(\Omega)$ being finite. In this case a measurable and invariant function $h(P)$ must be constant almost everywhere on Ω, and (11) yields, $h = 1, f^* =$ const.

$$f^* = \frac{\int_\Omega f\,dm}{m(\Omega)}. \tag{12}$$

A classical case of quasiergodicity is the following. Let $x_1, x_2, \ldots, x_n$ be the cartesian coördinates of a point of the n-dimensional Euclidean space. If two points, the corresponding coördinates of which differ by integer multiples of 2π, are identified, we obtain the points of the n-dimensional torus, where $x_1, x_2, \ldots, x_n$ now represent the angular coördinates of P. We then define P_t by the coördinates

$$x_1 + a_1t,\ x_2 + a_2t,\ \ldots,\ x_n + a_nt,$$

where the numbers $a_1, a_2, \ldots, a_n$ are linearly independent. This flow on the torus possesses an invariant measure m. From the Fourier series

$$f \sim \sum a_{v_1,v_2,\ldots v_n} \exp i(v_1x_1 + v_2x_2 + \ldots + v_nx_n)$$

of a function with finite $||f||$ we infer

$$U_tf \sim \sum b_{v_1,v_2,\ldots v_n} \exp. i(v_1x_1 + \ldots + v_nx_n),$$

$$b_{v_1,v_2,\ldots v_n} = a_{v_1v_2\ldots v_n} \exp. it(v_1a_1 + \ldots + v_na_n),$$

which immediately shows that an invariant function must be constant almost everywhere.

The above case has been treated in a well-known paper by Weyl[4] who showed that $\lim_{T=\infty} \frac{1}{T}\int_0^T f(P_t)dt$ exists for all points P, if $f(P)$ is continuous everywhere, or if f is the characteristic function of an open point set. This perfect uniformity of the flow, however, is a special feature of the above case. In the general quasiergodic case the exceptional points of Birkhoff's theorem exist.

4. *On Birkhoff's Time Average Theorem.*—Professor Birkhoff's proof of the existence of the time average in the sense of actual convergence is

based upon entirely different considerations of very general nature. In Birkhoff's paper, however, their generality is somewhat obscured by the usage of a manifold of section. The results appear in the following general form, if the proof is arranged in the following way.

Let $P \longrightarrow P_1$ be a single one to one transformation of Ω into itself which preserves the measure m. $m(\Omega)$ is supposed to be finite. We set

$$\phi_n(P) = \frac{1}{n}\,(f(P) + f(P_1) + \ldots + f(P_{n-1})),$$

where $f(P)$ is a bounded and measurable function, and where $P_2 = (P_1)_1$, ... The main point of Birkhoff's method is the

Lemma. If

$$\limsup_{n=\infty} \phi_n(P) \geqq \lambda$$

holds everywhere on an invariant and measurable set $A \subset \Omega$, then

$$\int_A f(P)dm \geqq \lambda m(A).$$

The proof is almost exactly the same as in Birkhoff's paper. From the lemma we easily infer the inequality

$$\int_\Omega \limsup_{n=\infty} \phi_n(P)dm \leqq \int_\Omega f(P)dm.$$

In replacing f by $1-f$ we obtain

$$\int_\Omega \liminf_{n=\infty} \phi_n(P)dm \geqq \int_\Omega f(P)dm.$$

This yields immediately the

THEOREM. The limit

$$\lim_{n=\infty} \phi_n(P)$$

exists almost everywhere on Ω.

Let, now, $P \longrightarrow P_t$, $(P_t)_s = P_{t+s}$ be again a group of the above kind and let $f(P)$ be a bounded and measurable function. In setting

$$g(P) = \int_0^1 f(P_t)dt$$

we infer by applying the theorem to $P \longrightarrow P_1$ and to $g(P)$,

$$\phi_n(P) = \frac{1}{n}\int_0^n f(P_t)dt.$$

Taking account to the boundedness of $f(P)$ we obtain immediately Birkhoff's general

TIME AVERAGE THEOREM. For any bounded and measurable function $f(P)$ the limit

$$\lim_{T=\infty} \frac{1}{T} \int_0^T f(P_t)dt$$

exists on Ω apart from a set of points P of measure zero.

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[1] Cf. B. O. Koopman, These PROCEEDINGS, 17, 315–318 (1931).

[2] Cf. J. v. Neumann, These PROCEEDINGS, 18, 70–82(1932).

[3] Cf. G. D. Birkhoff, These PROCEEDINGS, 17, 656–660(1931).

[4] Cf. H. Weyl, *Math. Annalen*, 77, 313–352 (1916).

SETS OF DISTINCT GROUP OPERATORS INVOLVING ALL THE PRODUCTS BUT NOT ALL THE SQUARES

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Suppose that $s_1, s_2, \ldots, s_k$ represent a set of distinct operators which obey the group laws when they are combined and include the product of every pair thereof, irrespective of the order of the factors, but not necessarily the square of any operator of the set. It is well known that a necessary and sufficient condition that such a set constitutes a group is that it contains also the square of each of its elements. In all cases the set generates a group G of order $g \geqq k$. In what follows it will be assumed that g is finite, and we shall determine all the possible groups of finite order which have the property that it is possible to find in each of them a set of distinct operators which generate the group and include the product of every pair of the set, irrespective of the order of the factors, but not the square of all of them. In other words, for each of these sets $k < g$.

When $k = 2$ the set is obviously composed of an arbitrary operator and the identity, and a necessary and sufficient condition is that G is cyclic and this operator is an arbitrary generator of this cyclic group. In this case $k = g$ when $g = 2$ and only then. When $k = 3$ the three operators s_1, s_2, s_3 must be commutative and if one of them is the identity, the other two are inverses of each other but are not otherwise restricted. Hence in this case G must again be cyclic, but its order is unrestricted except that it must exceed 2. A necessary and sufficient