is zero. *Then the A's are said to form a basis of the semi-group S.* If the A's are infinite in number, then the basis is said to be infinite, otherwise, finite.

We immediately see from the above that *the well-known factorization theorems in number-theory are included in theorems concerning basis systems of a semi-group.* For example, the theorem that a rational integer > 1 decomposes into prime factors uniquely may be stated in the form: The rational integers > 1 form, under multiplication, a quasi-group with a unique infinite basis. From the present view-point many factorization theorems in number theory in which the phrase is used, "aside from unit factors" now may be translated into theorems in which the behavior of the unit factors is definitely provided for. In connection with the decomposition of polynomials with coefficients in the rational field F, Gauss's lemma and related results may all be stated conveniently as theorems concerning the possible basis systems in the semi-group formed by multiplication of polynomials of this type.

[1] *Werke*, Bd. 3, halbband 1, pp. 263–273; H. B. Fine, *Bull. Amer. Math. Soc.* (1913).

[2] Algebra as a study of congruences with respect to functional moduli (Russian) Odessa (1913?), Chap. 8. I am indebted to Dr. A. E. Ross for an English translation of this.

[3] Note on a simple type of Algebra in which the cancellation law of addition does not hold. *Bull. Amer. Math. Soc.* (1935).

[4] *Werke*, Bd. 3, halbband 1, pp. 260–262.

[5] *Ann. Math.*, **24**, 263–264 (1923).

[6] Kronecker, *Werke*, Bd. 2, pp. 258–9; Van der Waerden, *Moderne Algebra*, Erster Teil, pp. 130–1 (1930).

FUNCTIONALITY IN COMBINATORY LOGIC*

By H. B. Curry

Department of Mathematics, The Pennsylvania State College

Communicated September 20, 1934

1. *Introduction.*—In an attempt to resolve the foundations of logic and mathematics into their elements, it has occurred to several persons that certain notions, ordinarily taken as primitive, could be analyzed into constituents of much simpler nature. Among such notions are, on the one hand, various processes of substitution, and the use of variables generally; and, on the other hand, the categories of logic—such as proposition, propositional function and the like—together with the intuitions by which we tell what entities belong to them.

For a theory concerned with an analysis of these notions I have proposed the name *combinatory logic* (*Amer. Jour. Math.*, **52**, 511 (1930)). This is a formal theory based on a primitive frame (i.e., set of primitive ideas, axioms and rules of procedure) of such great simplicity that even comparatively simple inferences, such as are ordinarily made by a substitution process can be decomposed into a large number of the elemental steps represented by the rules of procedure; it furthermore does not postulate any such notions as variable, although all the inferences ordinarily made by the use of variables can—when suitable definitions have been made, of course—be made as compound inferences within the system. The development of this theory, to the point where these statements relative to variables can be proved, is contained in a series of papers culminating in "Apparent Variables from the Standpoint of Combinatory Logic" (*Ann. Math.*, 2nd ser., **34**, 381–404 (1933)). To this paper, and to the earlier ones cited therein, the reader is referred for the details of the theory and for the notation.

The object of the present investigation is to obtain a similar analysis of the function concept. To make a more precise statement I must make the following explanations. We are concerned with statements of the form (using E. H. Moore's terminology) "f is a function on X to Y"; or (for functions of several variables) "f is a function on $X_1X_2 \ldots X_m$ to Y." The following type of question suggests itself: Suppose we have given certain entities $f_1, f_2, \ldots, f_n$ concerning each of which a statement of the above form has been made; suppose further that g is an entity derived from $f_1, \ldots, f_n$ by substitution or other such processes; then, what statement of the above form can we infer concerning g? We make inferences of the above form intuitively; what this paper asserts is that when certain formal constituents are added to the primitive frame of combinatory logic, then these inferences can be made abstractly within the extended system.

This formalization of the function notion is an important step toward the aims of combinatory logic (see *Amer. Jour. Math.*, **52,** 511). In the first place it gives *ipso facto* an analysis of the intuitions by which we classify entities into categories. For if the category of a given combination of entities is determinate, it is because the combination is constructed by substituting in certain functions entities which combine with those functions to give new entities of determinate character. It is only necessary to specify what the fundamental categories shall be, and to which ones the primitive entities belong. Thus the present theory provides an analysis of a class of inferences which are ordinarily made either tacitly or by virtue of complex rules which justify the inferences without giving a true analysis of them. In the second place this investigation has a bearing on the contradictions. For many of these contradictions appear to arise from applying the rules, appropriate to a certain category of entities, to an entity which seems to belong to that category but in reality does not do so. In such cases when we attempt to formalize the proof that the entity does belong to that category the contradiction dissolves. (See §§ 5 and 6 below.)

All the developments of this paper are independent of Ax. Π_0. (See reference above.)

2. *Axioms for Abstract Functionality.*—Let F denote an entity whose interpretation is such that FXY represents, in the notation of combinatory logic, the category of functions on X to Y, while the formula $\vdash FXYf$ represents the statement that f belongs to that category. This F may be called the *functionality relation*, it will be taken as a new primitive idea.

Let entities F_n be defined as follows:

Definition:
$$F_0 \equiv I$$
$$F_1 \equiv F$$
$$F_{n+1} \equiv [x_1, x_2, \ldots, x_n, y, z]\, F_n x_1 x_2 \ldots x_n (Fyz).$$

Then F_n has the same interpretation relative to functions of n variables that F has relative to functions of one variable; i.e., $F_n X_1 X_2 \ldots X_n Y$ represents the category of functions on $X_1, X_2, \ldots, X_n$ to Y, while the fact that f belongs to that category is expressed by the formula $\vdash F_n X_1 X_2 \ldots X_n Y f$.

The axioms for F are, then, as follows:

Axiom F.	$\vdash (x, y, z)(Fxyz \supset (u)(xu \supset y(zu))$
Axiom (FB).	$\vdash (x, y, z)F(Fxy)(F(Fzx)(Fzy))B$
Axiom (FC).	$\vdash (x, y, z)F(F_2yxz)(F_2xyz)C$
Axiom (FW).	$\vdash (x, y)F(F_2xxy)(Fxy)W$
Axiom (FK)	$\vdash (x, y)Fy(Fxy)K$
Axiom $(FP)_1$.	$\vdash (x, x', y)(P_1^*x'x \supset P_1^*(Fxy)(Fx'y))$
Axiom $(FP)_2$.	$\vdash (x, y, y')(P_1^*yy' \supset P_1^*(Fxy)(Fxy'))$
Axiom $(F\Pi)$.	$\vdash (x)F(Fx\text{Pr})\text{Pr}(P_1^*x)$

where Pr stands for proposition. The last axiom is of a somewhat more dubious character than the others.

3. *Consequences of These Axioms.*—The following are the principal theorems.

3.1. (Immediate consequence of the definition in §2.)

For $m, n = 0, 1, 2, 3, \ldots$

$$\vdash F_{m+n} = [x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, z]F_m x_1 x_2 \ldots x_m (F_n y_1 y_2 \ldots y_n).$$

I.e.: the two notions—"function (of $m + n$ variables) on $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ to z," and "function (of m variables) on $x_1, x_2, \ldots, x_m$ to functions (of n variables) on $y_1, y_2, \ldots, y_n$ to z" are identical.

3.2. (General substitution theorem, consequence of Axioms F to (FK), inclusive.) *If* —(1)$\mathfrak{X}$ *is a combination of variables* $t_1, t_2, \ldots, t_p, x_1, x_2, \ldots,$

x_m, ($p \geqq 0$, $m \geqq 0$), *and* $\alpha_1, \alpha_2, \ldots, \alpha_m, \delta$ *are combinations of constants and the variables* $t_1, t_2, \ldots, t_p$, *such that*

$$\vdash (t_1, t_2, \ldots, t_p) F_m \alpha_1 \alpha_2 \ldots \alpha_m \delta([x_1, x_2, \ldots, x_m]\mathfrak{X}),$$

(2) $\zeta_1, \zeta_2, \ldots, \zeta_m$ *are combinations of constants and the variables* $t_1, t_2, \ldots, t_p, y_1, y_2, \ldots, y_n$, *and* $\beta_1, \beta_2, \ldots, \beta_n$ *are combinations of constants and the variables* $t_1, t_2, \ldots, t_p$, *such that, for* $i = 1, 2, \ldots, m$,

$$\vdash (t_1, t_2, \ldots, t_p) F_n \beta_1 \beta_2 \ldots \beta_n \alpha_i([y_1, y_2, \ldots, y_n]\zeta_i),$$

(3) $\mathfrak{Y}$ *is the expression obtained from* $\mathfrak{X}$ *by substituting for each* x_i *wherever it occurs the corresponding* ζ_i; *then*

$$\vdash (t_1, t_2, \ldots, t_p) F_n \beta_1 \beta_2 \ldots \beta_n \delta([y_1, y_2, \ldots, y_n]\mathfrak{Y}).$$

This theorem represents the general result that if we have a function of m variables on $\alpha_1, \alpha_2, \ldots, \alpha_m$ to δ and if for each x_i we substitute a function of n variables on $\beta_1, \beta_2, \ldots, \beta_n$ to α_i, then the result is a function on $\beta_1, \beta_2, \ldots, \beta_n$ to δ. The result is valid when everything depends on certain parameters $t_1, t_2, \ldots, t_p$. It may be generalized by replacing hypothesis (2) by the following: (2′) *For* $i = 1, 2, 3, \ldots, m$, ζ_i *is a combination of constants, the* $t_1, t_2, \ldots, t_p$, *and a certain selection* $z_1^i, z_2^i, \ldots, z_{p_i}^i$ *of the variables* $y_1, y_2, \ldots, y_n$ *and* $\beta_1, \beta_2, \ldots, \beta_n$ *are combinations of constants and the* $t_1, t_2, \ldots, t_p$ *such that, for* $\gamma_k^i = \beta_j$ *whenever* $z_k^i = y_j$, *we have*

$$\vdash (t_1, t_2, \ldots, t_p) F_{p_i} \gamma_1^i \gamma_2^i \ldots \gamma_{p_i}^i \alpha_i([z_1^i, z_2^i, \ldots, z_{p_i}^i]\zeta_i).$$

This takes care of the case where the ζ_i do not involve all the $y_1, y_2, \ldots, y_n$.

3.3. (Special consequences of Axioms F to (FK), inclusive.) *The following formulas hold:*

$$\vdash (x) FxxI \qquad (1)$$

$$\vdash (x, y, u_1, u_2, \ldots, u_n) F(Fxy)(F(F_n u_1 u_2 \ldots u_n x)(F_n u_1 u_2 \ldots u_n y)) B_n. \qquad (2)$$

3.4. (General consequence of Axioms $(FP)_1$, $(FP)_2$, F.) *If* $X, \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_1', \alpha_2', \ldots, \alpha_n', \beta, \beta'$ *are entities such that*

$$\vdash (x) \alpha' x \supset \alpha_i x \qquad (i = 1, 2, \ldots, n),$$

and $\vdash (x) \beta x \supset \beta' x$;

then $\vdash (x)(F_n \alpha_1 \alpha_2 \ldots \alpha_n \beta x \supset F_n \alpha_1' \alpha_2' \ldots \alpha_n' \beta' x)$.

3.5. (General consequence of Axiom ($F\Pi$), involves also Axioms F – (FK), inclusive.) *If we make the definition*

$$\Xi_n \equiv [x_1, x_2, \ldots, x_n, y](t_1)(x_1 t_1 \supset (t_2)(x_2 t_2 \supset \ldots \supset (t_n)(x_n t_n \supset y t_1 t_2 \ldots t_n) \ldots)), \qquad (1)$$

then

$$\vdash (x_1, x_2, \ldots, x_u) F(F_n x_1 x_2 \ldots x_n \mathrm{Pr}) \mathrm{Pr}(\Xi_n x_1 x_2 \ldots x_n), \qquad (2)$$

where Pr *stands for "proposition,"* (which, for the purposes of this paper is to be interpreted as that which is either true or false). By §3.3 (2) this may be generalized to give

$$\vdash (x_1, \ldots, x_m, y_1, \ldots, y_n)F(F_{m+n}x_1 \ldots x_m y_1 \ldots y_n \mathrm{Pr})(F_m x_1 \ldots x_m \mathrm{Pr})(B_m(\Xi_n y_1 \ldots y_n)). \quad (3)$$

Hence if we define, for any constants $X_1, X_2, \ldots, X_n$ and any combination $\mathfrak{X}$ of constants and the variables $x_1, x_2, \ldots, x_n$,

$$(x_1^{X_1}, x_2^{X_2}, \ldots, x_n^{X_n})\mathfrak{X} \equiv \Xi_n X_1 X_2 \ldots X_n([x_1, x_2, \ldots, x_n]\mathfrak{X}), \quad (4)$$

we have the following result: *If* $\mathfrak{Y}$ *is such that*

$$\vdash F_{m+n}X_1X_2 \ldots X_m Y_1 Y_2 \ldots Y_n \mathrm{Pr}([x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n]\mathfrak{Y}),$$

then

$$\vdash F_m X_1 X_2 \ldots X_m \mathrm{Pr}([x_1, x_2, \ldots, x_m](y_1^{Y_1}, y_2^{Y_2}, \ldots, y_n^{Y_n})\mathfrak{Y})). \quad (5)$$

The interesting case here is where all the X_i, Y_j are the same.

4. *Defined Functionality.*—It is evident that Axiom F expresses the characteristic property of F. Suppose we define

$$F' \equiv [x, y, z](u)(xu \supset y(zu)).$$

Then F' is a kind of functionality notion, analogous to F. If we take it to be the same as F, then the Axioms F to $(FP)_2$, inclusive (with their consequences, of course), become provable formulas, provided we assume

Axiom (*PB*). $\vdash (x, y, z)((x \supset y) \supset ((z \supset x) \supset (z \supset y)))$.
Axiom (*PC*). $\vdash (x, y, z)((x \supset (y \supset z)) \supset (y \supset (x \supset z)))$.
Axiom (*PW*). $\vdash (x, y)((x \supset (x \supset y)) \supset (x \supset y))$.
Axiom (*PK*). $\vdash (x, y)(x \supset (y \supset x))$.

(These axioms are intimately related to the corresponding ones for F'.)

As for Axiom ($F\Pi$), it can probably not be proved for F'. If it be assumed, we shall, with reasonable assumptions as to negation, etc., get a contradiction. This contradiction is considered in Appendix B of Russell's Principles of Mathematics (Cambridge, 1903). It gives rise to the construction of a function f, such that $\vdash F'\mathrm{PrPr}f$ while $P_1^*\mathrm{Pr}f$ is self-contradictory. On the other hand the formula $\vdash F\mathrm{PrPr}f$ is probably not provable, so that we may assume Axiom ($F\Pi$), as stated, for an abstract F.

Thus if we hold to the suggested interpretations F is a more restricted concept than F', and the assertion $\vdash FXYf$ says more about f than that fx belongs to Y for every x in X. For many purposes it is desirable to retain this more restricted F, and we obtain increased generality by so doing.

5. *The Russell Paradox.*—This paradox arises as follows: Let N stand for negation and Pr for proposition (which means, in this connection, whatever is either true or false). Let

$$\chi \equiv [f]N(ff) \quad (1)$$

(i.e., χ is the property of being non-self-predicable). Then

$$\vdash \chi\chi = N(\chi\chi). \tag{2}$$

Now all this is perfectly correct. In fact without variables we can define

$$\chi \equiv W(BN), \tag{3}$$

and then derive (2) simply by use of Rules W and B. But we do not have a paradox until we have, in addition to (2),

$$\vdash \mathrm{Pr}(\chi\chi), \tag{4}$$

from which it will follow that $\chi\chi$ is both true and false.

If we attempt to prove (4) we find that it follows from the assumptions

$$\vdash FE\mathrm{Pr}N, \tag{5}$$

$$\vdash (x)(Ex \supset FEEx), \tag{6}$$

where $E \equiv WQ$ is the category of all entities. Of these it is useless to deny (6); for it may be proved for F' and Axiom ($F\Pi$) is not involved. On the other hand (5) is, intuitively, not even plausible. If we replace it by $\vdash F\mathrm{Pr}\mathrm{Pr}N$ the proof of (4) fails. Of course (5) may be made more plausible by replacing N by an entity $\mathfrak{N}$ defined in terms of N and having an interpretation such as the property of being a false proposition, but we shall not get a contradiction unless we make an illegitimate assumption of type (5) for some one of the entities entering into $\mathfrak{N}$.

Under these circumstances there is, I contend, no paradox. The statement (2) is not in itself paradoxical.

A similar explanation applies to certain paradoxes involving relations.

6. *The Epimenides Paradox.*—This paradox is concerned with the application of logic, but deserves consideration none the less, since a logic which cannot be applied is useless. Suppose then we have an intuitive property ϕ and define

$$A \equiv (x)(\phi x \supset Nx).$$

The following are to be regarded as established by experiment:

$$\vdash \phi A \tag{1}$$

$$\vdash (x)(\phi x . x . \supset . x = A) \tag{2}$$

(i.e., all propositions having property ϕ except possibly A are false). This is a possible situation, although it may never have occurred; the question is what can logic say about it.

It is alleged that from (1) and (2) we have

$$\vdash A \supset NA \text{ and } \vdash NA \supset A.$$

Suppose we grant this for the sake of argument. To get a real contradiction we must also have

$$\vdash \mathrm{Pr}A \tag{3}$$

which would follow from

$$\vdash F\mathrm{PrPr}\phi. \qquad (4)$$

But (4), and for that matter also (3), is preposterous.

Other forms of the Epimenides paradox I shall not attempt to discuss. For Russell's propositional paradox see paragraph 4.

7. *Concluding Remarks.*—1. Since the preceding theory does not involve, except in illustrations, any special assumptions as to what the categories of logic shall be, it is not tied to any particular logical theory, new or old, but can serve as foundation for a great variety of them. Thus it is compatible with the engere Funktionenkalkül (Hilbert and Bernays, Grundlagen der Mathematik I, paragraph 4, Berlin (1934)), with the Principia Mathematica (when suitably modified), with Zermelo's Theory of abstract sets (*Mathematische Annalen*, **65**, 261–281 (1908)), or with Heyting's quasi-intuitionistic theories (Berliner Sitzungs berichte, 1930). In fact it would seem to be fundamental to any theory in which distinction of categories is made at all. I am in general agreement with those who believe that any satisfactory logical theory must make such distinction.

2. In relation to Zermelo's theory we may redefine definiteness thus: f shall be definite over M when and only when $\vdash FM\mathrm{Pr}f$. The objections to the idea of definiteness will presumably not then apply. This should be compared with Fraenkel's definition (see, e.g., *Math. Zeit.*, **22**, 254 (1925) —Fraenkel's formulation is equivalent to a definition of definiteness, although he does not phrase it that way), which however rests on a special *ad hoc* definition of function, instead of on a general one, as here.

3. It has been objected that there is no assurance that this theory does not lead to other contradictions. True. However, the theory is no different from any other logical theory in that respect; indeed it is questionable if we shall ever have such assurance, other than that derived from empirical considerations, for any logical theory sufficiently powerful to be of any use. There is, therefore, little to be gained by adopting a Fabian policy in regard to these contradictions. Our friends the physicists have ceased to search for a theory of which they could be sure beforehand that it would explain the universe. We shall do better if we likewise make bold hypotheses which can be modified later, as further research shows their inadequacy. At several points I have made such hypotheses for the sake of simplicity, and propose this theory for examination and, possibly, later modification.

* The results of this investigation were presented in part to the American Mathematical Society on September 9, 1930. Research on this topic was continued while the author was National Research Council Fellow at the University of Chicago in 1931–32. Minor revisions have been made since.