# SPHEROIDAL FUNCTIONS

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If the wave equation

$$\nabla^2 V + k^2 V = 0$$

be separated in the coördinates of the elliptic cylinder, or the prolate or oblate spheroids,[1] it is observed that both radial and angular functions satisfy an equation of the type

$$(1 - z^2)w'' - 2(a + 1)zw' + (b - c^2z^2)w = 0. \qquad (1)$$

In the case of the elliptic cylinder, the parameter $a$ has the value $-{}^1/_2$ and (1) is the Mathieu equation in algebraic form, whereas in the spheroidal case, $a$ is a positive integer. Of the two independent solutions of (1), one at least must be finite at the poles $\pm 1$ if the usual requirements of a physical problem are to be satisfied, and this condition restricts the parameter $b$ to a discrete set of characteristic values in terms of $a$ and $c$. The solutions of (1) corresponding to characteristic values of $b$ will be termed spheroidal functions, and it is the object of the present note to define these functions in a manner most appropriate to physical applications and to state some of their most important properties. The details of the proofs, further properties of the functions of the second kind and an investigation of the conditions of convergence will appear elsewhere at a later date.

Equation (1) is characterized by an irregular singularity of the second species at infinity and by regular singularities at $\pm 1$, at each of which the exponents are 0 and $-a$. It is not the most general representative of this type since two of the possible five irreducible constants have been placed equal to zero. In the neighborhood of the point at infinity it is desirable to establish solutions which in the limit reduce to spherical waves. Such functions, however, are not valid in the region of small values of $z$ and it is necessary to define solutions appropriate to this domain and to determine their analytic connection with the functions suitable to the neighborhood of infinity. In what follows $a$ and $c$ may have any real or complex values provided $R(a) > -1$.

If in (1) we set $c = 0$ the equation admits of polynomial solutions when $b = l(l + 2a + 1)$ and $l$ is any positive integer inclusive of zero. These polynomials are the Gegenbauer functions $C_l^{a+1/2}(z)$, but since their numerical values have not been tabulated, it appears advantageous to define a slightly modified function which for integral values of $a$ is simply related to the associated Legendre functions. Let

$$T_l^a\ (z) = \frac{\Gamma(2a)}{2^{a-1}\Gamma(a)}\, C_l^{a+1/2}(z), \tag{2}$$

or as a hypergeometric function

$$T_l^a\ (z) = \frac{\Gamma(2a + l + 1)}{2^a\Gamma(a + 1)\Gamma(l + 1)}\, F\left(2a + l + 1,\ -l,\ a + 1,\ \frac{1 - z}{2}\right). \tag{3}$$

Then if $a = m$, an integer,

$$T_l^m(z) = \frac{d^m}{dz^m}\, P_{l+m}\ (z) = (1 - z^2)^{-m/2}\, P_{l+m}^m(z). \tag{4}$$

One has furthermore

$$(2l + 2a + 1)zT_l^a = (l + 2a)T_{l-1}^a + (l + 1)T_{l+1}^a, \tag{5}$$

$$\int_{-1}^{1}(1 - z^2)^a\, T_l^a(z)T_k^a(z)dz = \frac{2\Gamma(l + 2a + 1)}{(2l + 2a + 1)\Gamma(l + 1)}\, \delta_{lk}. \tag{6}$$

The solution of (1) which shall be called of the first kind, valid in the neighborhood of the cordinary point $z = 0$, is now obtained as an expansion in terms of the functions $T_n^a$. If $c \neq 0$ the characteristic value $b$ is of the form $b = l(l + 2a + 1) + \epsilon_l(c)$, where $\epsilon_l(c)$ vanishes with $c$ and is to be determined such that the expansion converges at $z = \pm 1$. Let

$$w_l(a, c;\ z) = \sum_n d_n^l T_n^a(z)$$

be a solution of the type required. The coefficients $d_n^l$ satisfy the recursion formula

$$\frac{(n + 2a + 2)(n + 2a + 1)}{(2n + 2a + 5)(2n + 2a + 3)}\, c^2 d_{n+2}^l + \frac{n(n - 1)}{(2n + 2a - 1)(2n + 2a - 3)}\, c^2 d_{n-2}^l + \left[\frac{2n^2 + 2n(2a + 1) + 2a - 1}{(2n + 2a - 1)(2n + 2a + 3)}\, c^2 + n(n + 2a + 1) - b\right] d_n^l = 0, \tag{7}$$

and since there is both an even and an odd series, $n$ may have the values $n = 0, 2, 4 \ldots$ or $n = 1, 3, 5 \ldots$. It will be observed that as $c$ vanishes, there remains only

$$[n(n + 2a + 1) - l(l + 2a + 1)]d_n^l = 0,$$

and hence all coefficients must vanish with $c$ with the exception of $d_l^l$ which remains finite. It is evident, furthermore, that the even series in $n$ is associated with even values of the index $l$, and the odd series in $n$ with odd values of $l$. We define:

$$Se^1_{a,l}(c, z) = \sum_n{}' d^l_n T^a_n(z), \tag{8}$$

wherein it is assumed that there has been assigned to $b$ a characteristic value such that the expansions converge at $z = \pm 1$. The prime indicates that the summation is over all even values of $n$ if $l$ is even and over all odd values if $l$ is odd. The question of normalization is deferred until certain other functions have been defined as follows.

If $z \gg 1$, (1) is satisfied approximately by

$$w \simeq (cz)^{-a-1/2} Z_p(cz), \qquad [p^2 = b + (a + 1/2)^2],$$

where $Z_p(cz)$ is any solution of the Bessel equation. One is led therefore to seek expansions valid in the neighborhood of $z = \infty$ of the type

$$w_l(a, c; z) = (cz)^{-a-1/2} \sum_n a^l_n Z_{n+a+1/2}(cz).$$

It may be verified that the coefficients $a^l_n$ satisfy the recursion formula

$$\frac{(n+1)(n+2)}{(2n+2a+3)(2n+2a+5)} c^2 a^l_{n+2} + \frac{(n+2a-1)(n+2a)}{(2n+2a-3)(2n+2a-1)} c^2 a^l_{n-2} + \left[b - n(n+2a+1) - \frac{2n^2 + 2n(2a+1) + 2a - 1}{(2n+2a-1)(2n+2a+3)} c^2\right] a^l_n = 0, \tag{9}$$

and that again there is an even and an odd series in $n$ associated respectively with even and odd values of $l$. We define:

$$Re^1_{a,l}(c, z) = (cz)^{-a-1/2} \sum_n{}' a^l_n J_{n+a+1/2}(cz), \tag{10}$$

$$Re^2_{a,l}(c, z) = (cz)^{-a-1/2} \sum_n{}' a^l_n N_{n+a+1/2}(cz), \tag{11}$$

where $N$ is the Neumann function.

$$Re^3_{a,l} = Re^1_{a,l} + i\,Re^2_a; \qquad Re^4_{a,l} = Re^1_{a,l} - i\,Re^2_{a,l}. \tag{12}$$

The normalization of the coefficients is now fixed on the basis of the asymptotic behavior of (10) and (11). For large values of the argument one has asymptotically for the even series

$$J_{2n+a+1/2}(cz) \simeq \sqrt{\frac{2}{\pi cz}} \cos\left(cz - \frac{2n+a+1}{2}\pi\right) = (-1)^{n-l} \sqrt{\frac{2}{\pi cz}} \sin\left(cz - \frac{2l+a}{2}\pi\right).$$

Let us normalize such that

$$\sum_{n=0}^{\infty} (-1)^{n-l} a^{2l}_{2n} = \sqrt{\frac{\pi}{2}}. \qquad (n, l = 0, 1, 2 \ldots). \tag{13}$$

The asymptotic expression for $Re^1_{2l}$ as $z$ approaches infinity is then

$$Re^1_{a,2l} \simeq (cz)^{-a-1} \sin\left(cz - \frac{2l+a}{2}\pi\right). \quad (l = 0, 1, 2 \ldots). \quad (14)$$

similarly for the odd series

$$\sum_{n=0}^{\infty} (-1)^{n-l} a^{2l+1}_{2n+1} = \sqrt{\frac{\pi}{2}}, \quad (n, l = 0, 1, 2 \ldots) \quad (15)$$

$$Re^1_{a,2l+1} \simeq (cz)^{-a-1} \sin\left(cz - \frac{2l+1+a}{2}\pi\right). \quad (l = 0, 1, 2 \ldots). \quad (16)$$

For the functions of the second kind

$$Re^2_{a,l} \simeq (cz)^{-a-1} \sin\left(cz - \frac{l+1+a}{2}\pi\right), \quad (l = 0, 1, 2 \ldots). \quad (17)$$

In order to deduce the analytic connections between the various functions, use is made of certain integral representations of the solution of (1). If a Laplace transformation of the type

$$w(z) = \int_C e^{iczt}(1 - t^2)^a u(t) dt \quad (18)$$

be introduced into (1), it is found that if (18) is to be a solution, $u(t)$ must itself satisfy (1) and the contour must be such that the bilinear concomitant

$$\left| e^{iczt}(1 - t^2)^{a+1}\left[iczu - \frac{du}{dt}\right]\right|_C \quad (19)$$

vanishes identically. In particular one may take $u(t) = Se^1_{a,l}(t)$ and choose the section of the real axis between $-1$ and $+1$ as the path of integration. Then with the aid of the integral

$$(cz)^{-a-1/2} J_{n+a+1/2}(cz) = \frac{(-i)^n \Gamma(n+1)}{\sqrt{2\pi}\,\Gamma(2a+n+1)} \int_{-1}^{1} e^{iczt} (1 - t^2)^a T^a_n(t) dt, \quad (20)$$

it is readily shown after multiplying both sides of (20) by $d^l_n$ and summing over $n$ that

$$k_l Re^1_{a,l} = \int_{-1}^{1} e^{iczt}(1 - t^2)^a Se^1_{a,l}(t) dt, \quad (21)$$

where $k_l$ is a proportionality factor, and

$$d^l_n = \frac{\Gamma(n+1)}{(i)^n \sqrt{2\pi}\,\Gamma(n+2a+1)} a^l_n k_l. \quad (22)$$

The coefficients $d_n^l$ are now normalized such that the functions $Se_{a,l}^1\ (z)$ are of unit magnitude at $z = \pm 1$. This is accomplished with the aid of (3), and one finds that if the proportionality factor $k_l$ be fixed so that

$$k_l = (i)^l 2^{a+1}\Gamma(a + 1), \tag{23}$$

then

$$Se_{a,l}^1(1) = 1, \qquad Se_{a,l}^1(-1) = (-1)^l, \tag{24}$$

and in virtue of (22), (13) and (15) also

$$\sum_n{}' \frac{\Gamma(n + 2a + 1)}{\Gamma(n + 1)}\, d_n^l = 2^a\Gamma(a + 1). \tag{25}$$

Since $Se_{a,l}^1$ and $Re_{a,l}^1$ are both integral functions of $z$, and both solutions of the same equation constructed with the same parameters, they can differ only by a factor independent of $z$.

$$Se_{a,l}^1 = \lambda_l k_l Re_l^1, \tag{26}$$

or

$$Se_{a,l}^1(z) = \lambda_l \int_{-1}^{1} e^{icst}(1 - t^2)^a Se_{a,l}^1(t)dt. \tag{27}$$

Of the many other contour integral expressions for the solutions of (1) only the following will be mentioned at this time.

$$k_l Re_{a,l}^3 = 2\int_{i\infty}^{1} e^{icst}(1 - t^2)^a Se_{a,l}^1(t)dt, \tag{28}$$

$$k_l Re_{a,l}^4 = 2\int_{-1}^{i\infty} e^{icst}(1 - t^2)^a Se_{a,l}^1(t)dt, \tag{29}$$

provided $R(z) > 1$.

The functions $Se_{a,l}^1$ are by no means the only solutions of (1) which may be constructed with the polynomials $T_n^a$ provided one withdraws the requirements of finiteness and single-valuedness at $z = \pm 1$. For if one place $w = (1 - z^2)^a u(z)$ it may be verified that $u(z)$ must satisfy

$$(1 - z^2)u'' - 2(1 - a)zu' + (b + 2a - c^2z^2)u = 0, \tag{30}$$

and if $c = 0$, (30) admits of polynomial solutions of the type $T^{-a}(z)$ when $b$ is given one of the characteristic values $b = l(l - 2a + 1) - 2a$. In analogy with what has preceded, we define a new set of functions

$$So_{a,l}^1 = (1 - z^2)^{-a} \sum_n{}' f_n^l T_n^{-a}(z). \tag{31}$$

The coefficients $f_n^l$ satisfy the recursion formula (7) if we replace there $a$ by $-a$ and $b$ by $b + 2a$. The nature of the functions $Se_{a,l}^1$ and $So_{a,l}^1$ is clarified if one makes the transformation $z = \cos\theta$. The corresponding Fourier expansions are then

$$Se^1_{a,l}(c, \theta) = \sum_n{}' D^l_n \cos n\theta, \qquad (32)$$

$$So^1_{a,l}(c, \theta) = (\sin \theta)^{-2a-1} \sum_n{}' F^l_{n+1} \sin (n+1)\theta. \qquad (33)$$

Associated with the $So^1_{a,l}$ are the functions $Ro^1_{a,l}$ which by analogy with the $Re^1_{a,l}$ are most readily defined by

$$(-1)^a k_l Ro^1_{a,l}(c, z) = (z^2 - 1)^{-a} \int_{-1}^{1} e^{iczt} So^1_{a,l}(t)dt, \qquad (34)$$

$$Ro^1_{a,l} = (z^2 - 1)^{-a}(cz)^{a-1/2} \sum{}' g^l_n J_{n-a+1/2}(cz), \qquad (35)$$

$$\frac{(i)^n\sqrt{2\pi}\Gamma(n - 2a + 1)}{\Gamma(n + 1)} f^l_n = k_l g^l_n. \qquad (36)$$

The determination of the characteristic values $b$ and the expansion coefficients as functions of the parameters $a$ and $c$ is essential to the complete definition of the functions, and a detailed account of this portion of the investigation will be given elsewhere.

[1] Bateman, *Partial Differential Equations of Mathematical Physics*, p. 440 et seq. For a rather complete account of previous work on the subject and a bibliography, see Strutt, *Lame'sche-Mathieusche-und verwandte Funktionen in Physik und Technik*, in the collection *Ergebnisse der Mathematik*, Springer, 1932.

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# ADDITION FORMULAE FOR SPHEROIDAL FUNCTIONS

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The functions developed by Stratton[1] in the preceding paper are of considerable importance in the study of wave motion in elliptic cylinder and in spheroidal coördinates. By their means a large number of diffraction problems can be studied: the scattering of waves from a thin strip, from a rod or a disc, the diffraction of waves through a slit or through a circular aperture, the scattering of electron waves from a diatomic molecule, etc. Before these problems can be solved, however, a number of addition formulae must be obtained, relating the spheroidal functions to the other known solutions of the wave equation. Some of these formulae are developed below.

1. *Elliptic Cylinder Coördinates.*—In the elliptic cylinder coördinates, $x = (d/2)\cos\varphi \cosh\mu$, $y = (d/2)\sin\varphi \sinh\mu$, the solutions of the wave equation which are everywhere finite are,