GENERAL ERGODIC THEOREMS

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1. *Generalities.*—By an "ergodic theorem," we mean a theorem asserting the convergence of the means of the transforms T_g of an element of a linear space E, under a group or semi-group G of linear operators T_g, to a limit fix-point. The present note is a sketch of a new approach to ergodic theorems over general semi-groups; details will be given elsewhere.

The first problem is to define convergence for general semi-groups. This can be solved by a new definition of convergence with respect to any transitive ordering, which specializes to Moore-Smith convergence.[1] The definition is independent of the existence of any measure or topology on the semi-group. In this it resembles von Neumann's definition of the mean of an almost periodic function on a group, which it contains.[2]

For definiteness, we shall assume that E is a Banach space, and the norms of the T_g are uniformly bounded; we shall not assume that G is commutative

Ergodicity cannot be proved without further restrictions. In fact, there is a well-known cyclic group of linear operators of norm one on the space (L), under which the means of the transforms of most elements do not converge at all.

Accordingly, let us define an "ergodic element," as an element the means of whose transforms converge to a fix-point. It can be shown that the set of ergodic elements is always a closed subspace, "invariant" in the sense of being transformed within itself by every T_g. This permits one to treat its quotient-space by similar methods.

Also, the closed convex hull of the transforms of any ergodic element contains a *unique* fix-point. In fact,

THEOREM 1: *All* the elements of E (or an invariant subspace) are ergodic, if and only if the closed convex hull of the transforms of every element contains one and only one fix-point.

2. *The Minimal Method.*—If G is Abelian, then the means of the transforms of any have the property of Moore-Smith: any two have a common "successor." Hence in this case (the only one treated in the literature[3]), there can be only one fix-point. But one can show[1]

THEOREM 2: If E is "uniformly convex," and the T_g are of norm at most one, then the closed convex hull of the transforms of any element contains *at least* one fix-point, namely, the unique point nearest the origin.

COROLLARY: If G is Abelian in Theorem 2, then every element of E is ergodic.

However, in the non-commutative case, we can have the new phenomenon of more than one fix-point. In fact, there is a two-element semi-group of linear contractions of a uniformly convex plane, under which the closed convex hull of the transforms of some elements include infinitely many fix-points.

3. *The Method of Nearly Invariant Measures.*—Our purely algebraic definition of "convergence" for the means of the transforms of an element is equivalent in the cases treated in the literature to the more familiar notion of convergence with respect to a sequence of measures.

To establish this equivalence, we define a group or semi-group as "ergodic," when it has a sequence of measures $m_n(V)$ with the properties (i) $m_n(G) = 1$ for all n and (ii) given $T_a \in G$, for sufficiently large n, $m_n(T_a V)$ and $m_n(VT_a)$ differ arbitrarily little from $m_n(V)$ for all V. In other words, right- and left-multiplication change the measures increasingly little as $n \to \infty$.

The nth means in the cyclic case, and the means used in the case of r-parameter Abelian semi-groups, are examples of such means used in the literature. Group measures furnish other examples, and quite new examples can be constructed.

Then assuming as usual that the integrals (nth means)

$$\mu_n = \int_G \xi T_a dV_n$$

are defined (for which there are various sufficient conditions in special cases), one can assert

THEOREM 3: An element ξ is "ergodic" with limit fix-point μ, if and only if $\mu_n \to \mu$, as $n \to \infty$.

This shows that our new definition of ergodicity is a true generalization of the usual concept: it is equivalent to the latter wherever the latter is defined. It follows *a posteriori*, that the usual concept of ergodicity is really independent of the choice of an "ergodic sequence" of measure functions, and depends only on the *existence* of such a sequence.

But applying the usual concept, and a modification of a construction of F. Riesz,[4] we can show that *if* the ξT_g lie in a *weakly compact* set, then ξ is ergodic in the usual sense—and hence by Theorem 3 in our sense. A host of special corollaries follow from this, including most known mean ergodic theorems, and new results in the theory of dependent probabilities.

4. *The Method of Invariant Means.*—One can weaken the assumption that a sequence of nearly invariant measures are definable on G, to the hypothesis that "means" are defined for bounded functions on G.

A linear functional $Mf(T_g)$ defined on the space of bounded real functions $f(T_g)$ on G is called a "mean" if it is non-negative for non-negative functions and gives to $f(T_g) = 1$ the mean $Mf = 1$. It is called right-invariant if $Mf(T_g T_a) = Mf(T_g)$, left-invariant if $Mf(T_a T_g) = Mf(T_g)$, for all $T_a \epsilon G$.

In the case we treat, if M is a mean, clearly $Mf(\xi T_g)$ exists for every $\xi \epsilon E$ and f in the adjoint E^* of E, and is a linear functional ϕ_ξ on E^* for each ξ. If M is right-invariant, then ϕ_ξ is a fix-point for all the second adjoints T_g^{**} of the T_g. If ϕ_ξ is in E, it is in the closed convex hull of the transforms of ξ.[5] If the closed convex hull of the transforms of ξ is weakly bicompact, which is the case if E is reflexive, then ϕ_ξ will be in E.

On the other hand, if M is left-invariant, and if the closed convex hull of the transforms of ξ contains a fix-point μ, then $f(\mu) = Mf(\xi T_g)$, so that μ is unique. Thus in this case, ξ is ergodic if and only if the closed convex hull of its transforms contains *at least* one fix-point. Hence if a right-invariant mean also exists, ξ will be ergodic if the closed convex hull of its transforms is weakly bicompact. So

THEOREM 5: If right- and left-invariant means exist for G, and if the closed convex hull of the transforms of ξ is weakly bicompact, then ξ is ergodic.

But if G is Abelian, a construction of Banach shows that such means always exist (they also exist for certain other G)[6] giving an obvious corollary. We also note that if such means exist, the transformation ϕ_ξ on E to E^{**}, which is always linear, defines a projection of the set of ergodic elements to the set of fix-points which is commutative with every T_g; in fact, this follows from our original definition.

[1] Cf. the preceding note, "An Ergodic Theorem for General Semi-Groups," §1.

[2] J. von Neumann, "Almost Periodic Functions on Groups," *Trans. Am. Math. Soc.*, **36**, 456–89 (1934).

[3] N. Dunford, "A Mean Ergodic Theorem," *Duke Jour.*, **5**, 635–646 (1939), and papers (including Wiener's) cited there.

[4] F. Riesz, "Some Mean Ergodic Theorems," *Jour. Lond. Math. Soc.*, **13**, 274–278 (1938).

[5] T. H. Hildebrandt, "On Bounded Linear Functional Operations," *Trans. Am. Math. Soc.*, **36**, 875 (1934); S. Mazur, "Über konvexe Mengen in linearen normierten Raumen," *Studia Math.*, **4**, 70 (1933).

[6] J. von Neumann, "Zur allgemeinen Theorie des Masses," *Fund. Math.*, **13**, 73 (1929).