Under the assumption that each μ_k, k = 0, 1, ..., *is finite, the following Morse relations hold:*

$$\begin{aligned} \mu_0 &\geq 1 \\ \mu_1 - \mu_0 &\geq -1 \\ &\vdots \\ \mu_n - \mu_{n-1} + \ldots + (-1)^n \mu_0 &\geq (-1)^n \\ &\vdots \end{aligned}$$

11. An example for the application of the theory is a Jordan curve which bounds a small strip around a pair of conjugate cuts on a torus. This curve, as pointed out by Courant, bounds at least two minimal surfaces which are proper relative minima. An example with two boundaries, which has not been taken up in this note, is the classical case of two circles with line of centers perpendicular to the planes of the circles.

[1] This note was read at the Sept., 1938, meeting of the American Mathematical Society, New York City.

[2] This is an essential step in the methods of Douglas, Radó, Courant.

[3] This follows from the lower semi-continuity of the Dirichlet functional and the equicontinuity of the boundary values of all surfaces in $\mathfrak{P}_N$. Cf. Courant, "Plateau's Problem and Dirichlet's Principle," *Ann. Math.*, **38**, 679–724 (1937), esp. pp. 690–692.

[4] The Lemma remains true without this restriction.

[5] Morse, "Analysis in the Large," notes of the Institute for Adv. Study, 1936–1937; "Functional Topology and Abstract Variational Theory," *Ann. Math.*, **38**, 386–448 (1937); and "The Calculus of Variations in the Large," *Amer. Math. Soc. Coll. Publ.*, **18**.

[6] By a bloc of minimal surfaces is meant a maximal connected set of minimal surfaces $\mathfrak{x}$ bounded by Γ for which $D[\mathfrak{x}]$ = constant. It is very unlikely that a bloc consists of more than one surface.

CONSISTENCY-PROOF FOR THE GENERALIZED CONTINUUM-HYPOTHESIS[1]

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If M is an arbitrary domain of things in which a binary relation ϵ is defined, call "*propositional function over M*" any expression φ containing (besides brackets) only the following symbols: 1. Variables x, y .... whose range is M. 2. Symbols $a_1 \ldots a_n$ denoting[2] individual elements of M (referred to in the sequel as "*the constants of* φ"). 3. ϵ. 4. $\sim$ (not), $\vee$ (or). 5. Quantifiers for the above variables x, y ...* Denote by M' the set of all subsets of M defined by prop. funct. $\varphi(x)$ over M. Call a function f with s variables a "*function in M*" if for any elements $x_1 \ldots x_s$ of M

$f(x_1 \dots x_s)$ is defined and is an element of M. If $\varphi(x)$ is a prop. funct. over M with the following normal form:

$$(x_1 \dots x_n)\ (\exists y_1 \dots y_m)\ (z_1 \dots z_k)\ (\exists u_1 \dots u_e) \dots . \\ L(xx_1 \dots x_n y_1 \dots y_m z_1 \dots z_k u_1 \dots u_e \dots .)$$

(L containing no more quantifiers) and if $a \in M$, then call "*Skolem-functions for φ and a*" any functions $f_1 \dots f_m\ g_1 \dots g_e \dots$ in M with resp. $n \dots n$, $n + k \dots n + k \dots$ variables such that for any elements $x_1 \dots x_n z_1 \dots z_k \dots$ of M the following is true:

$$L(ax_1 \dots x_n\, f_1(x_1 \dots x_n) \dots f_m(x_1 \dots x_n)\, z_1 \dots z_k \\ g_1(x_1 \dots x_n z_1 \dots z_k) \dots g_e(x_1 \dots x_n z_1 \dots z_k) \dots .)$$

The proposition $\varphi(a)$ is then equivalent with the existence of Skolem-fnct. for φ and a.

Now define: $M_0 = \{\Lambda\}$, $M_{\alpha+1} = M_\alpha'$, $M_\beta = \sum\limits_{\alpha<\beta} M_\alpha$ for limit numbers β. Call a set x "*constructible*," if there exists an ordinal α such that $x \in M_\alpha$ and "*constructible of order α*" if $x \in M_{\alpha+1} - M_\alpha$. It follows immediately that: $M_\alpha \subset M_\beta$ *and* $M_\alpha \in M_\beta$ *for* $\alpha < \beta$ and that:

TH. 1. *$x \in y$ implies that the order of x is smaller than the order of y for any constr. sets x, y.*

It is easy to define a well-ordering of all constr. sets and to associate with each constr. set (of an arbitrary order α) a uniquely determined prop. fnct. $\varphi_\alpha(x)$ over M_α as its "*definition*" and furthermore to associate with each pair φ_α, a (consisting of a prop. fnct. φ_α over M_α and an element a of M_α for which $\varphi_\alpha(a)$ is true) uniquely determined "*designated Skolem-fnct. for φ_α, a.*"[3]

TH. 2. *Any constr. subset m of M_{ω_α} has an order $< \omega_{\alpha+1}$ (i.e., a constr. set, all of whose elements have orders $< \omega_\alpha$ has an order $< \omega_{\alpha+1}$).*

PROOF: Define a set K of constr. sets, a set O of ordinals and a set F of Skolem-fnct. by the following postulates I–VII:

I. $M_{\omega_\alpha} \subseteq K$ and $m \in K$.

II. If $x \in K$, the order of x belongs to O.

III. If $x \in K$, all constants occurring in the definition of x belong to K.

IV. If $\alpha \in O$ and $\varphi_\alpha(x)$ is a prop. fnct. over M_α all of whose constants belong to K then:

1. The subset of M_α defined by φ_α belongs to K.
2. For any $y \in K \cdot M_\alpha$ the design. Skolem-fnct. for φ_α and y or $\sim \varphi_\alpha$ and y (according as $\varphi_\alpha(y)$ or $\sim \varphi_\alpha(y)$) belong to F.

V. If $f \in F$, $x_1 \dots x_n \in K$ and $(x_1 \dots x_n)$ belongs to the domain of definition of f, then $f(x_1 \dots x_n) \in K$.

VI. If $x, y \in K$ and $x - y \neq \Lambda$ the first[4] element of $x - y$ belongs to K.

VII. No proper subsets of K, O, F satisfy I–VI.

TH. 3. *If* $x \neq y$ *and* $x, y \in K \cdot M_{\alpha+1}$, *then there exists a* $z \in K \cdot M_\alpha$ *such that* $z \in x - y$ or $z \in y - x$.[13]
(follows from VI and Th. 1.)

TH. 4.[5] $\overline{\overline{K + O + F}} = \aleph_\alpha$
since $\overline{\overline{M}}_{\omega_\alpha} = \aleph_\alpha$ *and* $K + O + F$ *is obtained from* $M_{\omega_\alpha} + \{m\}$ *by forming the closure with respect to the operations expressed by* II–VI.

Now denote by η the order type of O and by $\bar{\alpha}$ the ordinal corresponding to α in the similar mapping of O on the set of ordinals $< \eta$. Then we have:

TH. 5. *There exists a one to one mapping* x' *of* K *on* M_η *such that* $x \in y \equiv x' \in y'$ *for* $x, y \in K$ *and* $x' = x$ *for* $x \in M_{\omega_\alpha}$.

PROOF: The mapping x' (which will carry over the elements of order α of K exactly into all constr. sets of order $\bar{\alpha}$ for any $\alpha \in O$) is defined by transfinite induction on the order, i.e., we assume that for some $\alpha \in O$ an isomorphic[6] mapping f of $K \cdot M_\alpha$ on $M_{\bar{\alpha}}$[7] has been defined and prove that it can be extended to an isomorphic mapping g of $K \cdot M_{\alpha+1}$ on $M_{\bar{\alpha}+1}$[8] in the following way: At first those prop. fnct. over M_α whose constants belong to K (hence to $K \cdot M_\alpha$) can be mapped in a one to one manner on all prop. fnct. over $M_{\bar{\alpha}}$ by associating with a prop. fnct. φ_α over M_α having the constants $a_1 \ldots a_n$ the prop. fnct. $\varphi_{\bar{\alpha}}$ over $M_{\bar{\alpha}}$ obtained from φ_α by replacing a_i by a_i^1 and the quantifiers with the range M_α by quantifiers with the range $M_{\bar{\alpha}}$. Then we have:

TH. 6. $\varphi_\alpha(x) \quad \varphi_{\bar{\alpha}}(x^1)$ *for any* $x \in K \cdot M_\alpha$.

PROOF: If $\varphi_\alpha(x)$ is true, the design . Skolem-fnct. for φ_α and x exist, belong to F (by IV, 2) and are functions in $K \cdot M_\alpha$ (by $\overline{\text{V}}$). Hence they are carried over by the mapping f into functions in $M_{\bar{\alpha}}$ which are Skolem-functions for $\varphi_{\bar{\alpha}}$, x^1, because the mapping f is isomorphic with respect to ϵ. Hence $\varphi_\alpha(x) \supset \varphi_{\bar{\alpha}}(x^1)$.

$\sim\varphi_\alpha(x) \supset \sim\varphi_{\bar{\alpha}}(x^1)$ is proved in the same way.

Now any φ_α over M_α whose constants belong to K, defines an element of $K \cdot M_{\alpha+1}$ by IV, 1 and any element b of $K \cdot M_{\alpha+1}$ can be defined by such a φ_α (if $b \in M_{\alpha+1} - M_\alpha$ this follows by III, if $b \in M_\alpha$ then "$x \in b$" is such a φ_α). Hence the above mapping of the φ_α on the $\varphi_{\bar{\alpha}}$ gives a mapping g of all elements of $K \cdot M_{\alpha+1}$ on all elements of $M_{\bar{\alpha}+1}$ with the following properties:

A. *g is singlevalued*, because if φ_α, ψ_α define the same set, we have $\varphi_\alpha(x) \equiv \psi_\alpha(x)$ for $x \in M_\alpha \cdot K$, hence $\varphi_{\bar{\alpha}}(x^1) \equiv \psi_{\bar{\alpha}}(x^1)$ by Th. 6, i.e., $\varphi_{\bar{\alpha}}$ and $\psi_{\bar{\alpha}}$ also define the same set.

B. $x \in y \equiv x^1 \in g(y)$ *for* $x \in K \cdot M_\alpha$, $y \in K \cdot M_{\alpha+1}$.
(by Th. 6)

C. *g is one to one*, because if $x, y \in K \cdot M_{\alpha+1}$, $x \neq y$ then by Th. 3 there is a $z \in (x - y) + (y - x)$, $z \in K \cdot M_\alpha$, hence $z^1 \in [g(x) - g(y)] + [g(y) - g(x)]$ by B. Hence $g(x) \neq g(y)$.

D. *g is an extension of the mapping f, i.e.,* $g(x) = x^1$ *for* $x \in K \cdot M_\alpha$.

PROOF: For any $b \epsilon K \cdot M_\alpha$ a corresponding φ_α which defines it is $x \epsilon b$, hence $\varphi_{\bar{\alpha}}$ is $x \epsilon b^1$ hence $g(b) = b^1$.

E. *g maps* $K \cdot M_\alpha$ *exactly on* $M_{\bar{\alpha}}$ *(by D)*[9] *and therefore,* $K(M_{\alpha+1} - M_\alpha)$ *on* $M_{\bar{\alpha}+1} - M_{\bar{\alpha}}$ *by C.*

F. *g is isomorphic for* ϵ, *i.e.,* $g(x) \epsilon g(y) \equiv x \epsilon y$ *for any* $x, y \epsilon K \cdot M_{\alpha+1}$.

PROOF: If $x \epsilon K \cdot M_\alpha$, this follows from B and D, if $x \epsilon K \cdot (M_{\alpha+1} - M_\alpha)$ then $g(x) \epsilon M_{\bar{\alpha}+1} - M_{\bar{\alpha}}$ by E, hence both sides of the equivalence are false by Th. 1.

By D and F, g is the desired extension of f and hence the existence of an isomorphic mapping x' of K on M_η follows by complete induction. Furthermore since all ordinals $< \omega_\alpha$ belong to O (by I, II) we have $\bar{\beta} = \beta$ for $\beta < \omega_\alpha$ from which it follows easily that $x = x'$ for $x \epsilon M_{\omega_\alpha}$. This finishes the proof of Th. 5.

Now in order to prove Th. 2 consider the set m' corresponding to m in the isomorphic mapping of K on M_η. Its order is $< \eta < \omega_{\alpha+1}$, because $m' \epsilon M_\eta$ and $\bar{\bar{\eta}} = \bar{\bar{O}} \leq \aleph_\alpha$ by Th. 4. Since $x \epsilon m \equiv x' \epsilon m'$ for $x \epsilon K$, we have $x \epsilon m \equiv x \epsilon m'$ for $x \epsilon M_{\omega_\alpha}$ by Th. 5. Since furthermore $m \subseteq M_{\omega_\alpha}$ it follows that $m = m' \cdot M_{\omega_\alpha}$, i.e., m is an intersection of two sets of order $< \omega_{\alpha+1}$, which implies trivially that it has an order $< \omega_{\alpha+1}$.

TH. 7. M_{ω_ω} *considered as a model for set-theory satisfies all axioms of Zermelo*[10] *except perhaps the axiom of choice and* M_Ω *(*Ω *being the first inaccessible number) satisfies in addition the axiom of substitution, if in both cases "definite Eigenschaft" resp. "definite Relation" is identified with "prop. fnct. over the class of all sets" (with one resp. two free variables).*

Sketch of proof for M_{ω_ω}: ax. I, II are trivial, ax. VII is satisfied by $Z = M_\omega$, ax. III–V have the form $(\exists x)(u)[u \epsilon x \equiv \varphi(u)]$, where the φ are certain prop. fnct. over M_{ω_ω}. Hence, by def. of $M_{\alpha+1}$ there exist sets x in $M_{\omega_\omega+1}$ satisfying the axioms. But from Th. 1 and Th. 2 it follows easily, that the order of x is smaller than ω_ω for the particular φ under consideration, so that there exist sets x in the model satisfying the axioms.

For M_Ω ax. I–V and VII are proved in exactly the same way and the axiom of subst. is proved by the same method as ax. III–V. Now denote by "A" the proposition "There exist no non-constructible sets"[11] by "R" the axiom of choice and by "C" the proposition "$2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for any ordinal α." Then we have:

TH. 8. $A \supset R$ *and* $A \supset C$.

$A \supset R$ follows because for the constr. sets a well-ordering can be defined and $A \supset C$ holds by Th.2, because $\bar{\bar{M}}_{\omega_\alpha} = \aleph_\alpha$.

Now the notion of "constr. set" can be defined and its theory developed in the formal systems of set theory themselves. In particular Th. 2 and, therefore, Th. 8 can be proved from the axioms of set theory. Denote the notion of "constr. set" relativized for a model M of set theory (i.e., defined in terms of the ϵ-relation of the model) by *constr.*$_M$, then we have:

TH. 9. *Any element of M_{ω_ω} (resp. M_Ω) is constr.$_{M_{\omega_\omega}}$ (resp. constr.$_{M_\Omega}$); in other words: A is true in the models M_{ω_ω} and M_Ω.*

The proof is based on the following two facts: 1. The operation M' (defined on p. 220) is absolute in the sense that the operation relativized for the Model M_{ω_ω}, applied to an $x \epsilon M_{\omega_\omega}$ gives the same result as the original operation (similarly for M_Ω). 2. The set N_α which has as elements all the M_β (for $\beta < \alpha$) is constr.$_{M_{\omega_\omega}}$ for $\alpha < \omega_\omega$ and constr.$_{M_\Omega}$ for $\alpha < \Omega$, as is easily seen by an induction on α. From Th. 9 and the provability (from the axioms of set theory) of Th. 8 it follows:

TH. 10. *R and C are true for the models M_{ω_ω} and M_Ω.*

The construction of M_{ω_ω} and M_Ω and the proof for Th. 7 and Th. 9 (therefore also for Th. 10) can (after certain slight modifications)[12] be accomplished in the resp. formal systems of set theory (without the axiom of choice), so that a contradiction derived from C, R, A and the other axioms would lead to a contradiction in set theory without C, R, A.

[1] This paper gives a sketch of the consistency proof for propositions 1, 2 of *Proc. Nat. Acad. Sci.*, **24**, 556 (1938), if T is Zermelo's system of axioms for set theory (*Math. Ann.*, **65**, 261) with or without axiom of substitution and if Zermelo's notion of "Definite Eigenschaft" is identified with "propositional function over the system of all sets." Cf. the first definition of this paper.

[2] It is assumed that for any element of M a symbol denoting it can be introduced.

[3] At first with each φ_α an equivalent normal form of the above type has to be associated, which can easily be done.

[4] In the well-ordering of the constr. sets.

[5] $\overline{\overline{m}}$ means "power of m."

[6] I.e., $x \epsilon y \equiv f(x) \epsilon f(y)$. In the following proof $f(x)$ is abbreviated by x^1.

[7] I.e., of the elements of order $<\alpha$ of K on the elements of order $<\bar{\alpha}$ of M_η.

[8] I.e., of the elements of order $\leq\alpha$ of K on the elements of order $\leq\bar{\alpha}$ of M_η.

[9] Because f maps $K \cdot M_\alpha$ on $M_{\bar{\alpha}}$ by induct. assumpt.

[10] Cf. *Math. Ann.*, **65**, 261 (1908).

[11] In order to give A an intuitive meaning, one has to understand by "sets" all objects obtained by building up the simplified hierarchy of types on an empty set of individuals (including types of arbitrary transfinite orders).

[12] In particular for the system without the axiom of substitution we have to consider instead of M_{ω_ω} an isomorphic image of it (with some other relation R instead of the ϵ-relation), because M_{ω_ω} contains sets of infinite type, whose existence cannot be proved without the axiom of subst. The same device is needed for proving the consistency of prop. 3, 4 of the paper quoted in footnote 1.

[13] Th. 3, 4, 5, are lemmas for the proof of Th. 2.

* Unless explicitly stated otherwise "prop. fnct." always means "propositional function with one free variable."