

In Theorem 12, we mean the (uniquely determined) products of 11 (a). In place of the following paragraphs, read: As knowing all $v_i^p \cdot u^p$ for all i uniquely determines u^p , and the correct \frown satisfies (11.8), we have found the correct \frown . (NOTE: Theorem 11 is not used.) The end of the first paragraph in 12 should read: Find $u^p \smile u^q$ over I_0 , then over R_1 , ($p \leq 2$), then for all p , by (5.12); then find \frown over I_0 , by Theorem 12. (We must know (11.15).) In Theorem 13, (a), add: $\phi\tau^p = 0$ if $O(\tau^p)$ is acyclic. Relation (14.9) follows directly from (14.4). After (25.7), add: $I^p \smile I^q = I^{p+q}$.

² *Augmentable* in Tucker, *Ann. Math.*, **34**, 191–243 (1933).

³ See Tucker, these PROCEEDINGS, **25**, 371–374 (July, 1939). The theory was developed independently by S. Lefschetz and myself.

⁴ This case of the theory is due to de Rham; see *Comm. Math. Helv.*, **4**, 151–157 (1933).

ON THE THEORY OF SPHERE-BUNDLES

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1. *Introduction.*—We give here a brief sketch of some new results in the theory of sphere-bundles;¹ in particular, further properties of the characteristic classes, a duality theorem, theorems on tangent and normal bundles to a manifold, and some examples. The results will be published later in book form.

2. *Fibre-Bundles.*—Let S_0 be a space, and G , a group of homeomorphisms of S_0 into itself. Then over any space K , the *base space*, with neighborhoods U_i , we may define a *fibre-bundle* $\mathfrak{B}(K)$ as follows. For $p \in U_i$, let $\xi_i(p)$ be a homeomorphism of S_0 into a set of points $S(p)$. Let $S(p) \cdot S(q) = 0$ if $p \neq q$. Let $\mathfrak{S}(K)$ be the space of all points on all $S(p)$, the *total space* (gefaserte Raum). Let $\xi_i(p, q)$ be the image of q in S_0 under $\xi_i(p)$. For $p \in U_i \cdot U_j$, set $\xi_{ij}(p, q) = \xi_i^{-1}(p, \xi_j(p, q))$; assume $\xi_{ij}(p) \in G$ for each p , and that it varies continuously with p . Then a topology is easily defined in \mathfrak{S} . The part $\mathfrak{S}(U_i)$ of \mathfrak{S} over U_i is a product $U_i \times S_0$.

If S_0 is a set of μ points, and G is the group of permutations, we obtain the covering spaces of K with μ sheets. If S_0 is a subgroup of a continuous group R_0 , and $G = S_0$, then the left (or right) cosets of S_0 form a space K (factor group if S_0 is normal); the total space is R_0 . If $S_0 = S_0^v$ is a v -sphere, and $G = G^{v+1}$, the orthogonal group, we have a *sphere-bundle*. If S_0 is a vector space, and G , the affine or orthogonal group, an equivalent theory is obtained.

3. *Particular Coördinate Systems.*—We use this section in the proof of the duality theorem. Let K be a complex with ordered vertices. We may use ξ_σ , defined over closed cells σ . (See TP.) We may choose them so $\xi_\sigma = \xi_{\sigma'}$ if σ and σ' have the same first vertex. Let P^σ be a small closed

region in K surrounding all $D_a(\sigma', \sigma')$, σ' of any dimension (see I, §2); set $Q' = K - P'$. Then if K^r is the r -dimensional part of K , and $e_1, \dots, e_{\nu+1}$ are the unit points of S_0^ν ,

$$\xi_{\sigma, \sigma'}(p, e_i) = e_i \quad (p \in Q^k \cdot \sigma \cdot \sigma', i = 1, \dots, \nu - k).$$

Note that $\phi_i(p) = \xi_{\sigma^r-1}(p, e_i)$ ($p \in$ any σ^{r-1} , $i = 1, \dots, \nu - r + 2$) defines orthogonal projections of K^{r-1} into \mathfrak{S} .

4. *Characteristic Classes.*—Choose $\phi_1, \dots, \phi_{\nu-r+2}$ as above over K^{r-1} ; then for each σ^r , studying these on $\partial\sigma^r$ gives $W^r \cdot \sigma^r$, which is an integer mod 2 if $r = 1$ or $r \leq \nu$ is even, and an integer otherwise. W^r is a cocycle whose class \mathbf{W}^r is an invariant of \mathfrak{B} ; the \mathbf{W}^r characterize \mathfrak{B} if $\nu \leq 1$ or $\dim(K) \leq 3$ (see TP). We may use a general type of subdivision of the polyhedron K in defining the \mathbf{W}^r .

If \mathfrak{B} is not orientable (TP, §4), and K' is obtained from K by replacing $[\sigma^r: \sigma^{r+1}]$ by $-[\sigma^r: \sigma^{r+1}]$ when ξ_{σ^r} and ξ_{σ^r+1} give opposite orientations to the $S(p)$ ($p \in \sigma^r$) (see TP, p. 793, footnote), then K' is locally isomorphic with K , and $\mathbf{W}' = \mathbf{W}^1$. We call K' the *complex associated with* \mathfrak{B} . The characteristic classes are taken in K' ; the theorems above hold still.

If f maps K_1 into K_2 , and $\mathfrak{B}_2(K_2)$ is defined, then a bundle $\mathfrak{B}_1(K_1)$ is defined (TP, §8), and $\mathbf{W}_1^r = f' \mathbf{W}_2^r$ (f' = dual of f).

If $\nu = 2$, $\dim(K) = 4$, and $\mathbf{W}^1 = 0$, $\mathbf{W}^2 = 0$, then an invariant characterizing \mathfrak{B} is obtained as follows. A triple $\phi = (\phi_1, \phi_2, \phi_3)$ of orthogonal projections of K^3 into $\mathfrak{S}(K^3)$ exists. Let $\Phi(p)$ ($p \in K^3$) map S_0 into $S(p)$ so that $\Phi(p, e_i) = e_i$ ($i = 1, 2, 3$). For each σ^4 , set

$$\Psi_{\sigma^4}(p) = \xi_{\sigma^4}^{-1}(p)\Phi(p) \quad (p \in \partial\sigma^4).$$

This maps $\partial\sigma^4$ into the orthogonal group G^3 ; as G^3 is homeomorphic with projective 3-space P^3 , this defines an integer $D^4 \cdot \sigma^4$, the degree of Ψ_{σ^4} . D^4 is a cocycle. If we identify two cocycles if they are cohomologous, or differ by a cocycle of the form $X^1 \smile X^1 \smile X^1 \smile X^1$ (X^1 a 1-cocycle), the class determined is the invariant.

The classes W^{2r+1} are determined from the others as follows (see I, §11):

$$W^{2r+1} = \frac{1}{2} \partial \omega W^{2r} \quad (\text{if } \nu \geq 2r).$$

5. *On Mappings into $G^{\nu+1}$.*—In the theorem just stated, and in the duality theorem, we need the following (and other more complicated) theorems. (a) Let f map σ^r into $G^{\nu+1}$ so that if $\phi(p) = f(p, e_1)$, then $\phi(p) = e_1$ in $\partial\sigma^r$; let ϕ be of degree α . Let ψ map σ^r into S_0^ν with the degree β , and let $\psi(p) = e_1(p \in \partial\sigma^r)$. Then $\theta(p) = f(p, \psi(p))$ is of degree $\alpha + \beta$. (b) Take f as before; then $\phi'(p) = f^{-1}(p, e_1)$ (the point of S_0 mapped into e_1 by $f(p)$) is of degree $-\alpha$. (Use (a).) (c) Let f map σ^r into $G^{\nu+1}$, let ϕ map $\partial\sigma^r$ into $S_0^{\nu-1}$ with the degree α , and suppose $f(p, \phi(p)) = e_1$ ($p \in$

$\partial\sigma^r$). Then $\psi(p) = f(p, e^{r+1})$ maps $\partial\sigma^r$ into the S_1^{r-1} orthogonal to e_1 with a degree $\equiv \alpha \pmod 2$.

6. *The Duality Theorem.*—Given bundles $\mathfrak{B}_1^\lambda(K)$ and $\mathfrak{B}_2^\mu(K)$, there is a uniquely determined bundle $\mathfrak{B}_3^\nu(K)$, their product ($\nu = \lambda + \mu + 1$; see TP, p. 796); thus if $M^m \subset M^n$, the tangent times the normal bundle gives the part of the tangent bundle of M^n over M^m . The formula for the characteristic classes of \mathfrak{B}_3^ν is

$$W_3^r = \sum_i W_1^i \smile W_2^{r-i}, \text{ reducing mod 2 if necessary.}$$

(See §4 and I, §12; we use $W^0 =$ sum of vertices.) The proof is very difficult if $r \geq 4$. We use the special $\xi_{1,\sigma}$ of §3 in \mathfrak{B}_1 , and $\xi_{2,\sigma}$ in \mathfrak{B}_2 , with a replaced by $a' < a$, so the P_1 and P_2 will be in "general position" (see I, §5). The projections into $\mathfrak{S}_3(K^{r-1})$ are defined successively over Q_1^0, Q_1^1, \dots . For each σ^r , they are now deformed in $\partial\sigma^r$ into a simpler position, except in each $A^i = P_1^i \cdot P_2^{r-1-i} \cdot \sigma^{r-1}$ ($\sigma^{r-1} =$ face of σ^r opposite first vertex of σ^r). The terms shown come from the A^i , two coming from A^{r-1} . The results of §5 and the products of I, §6, are needed.

REMARK. We do not know whether or not the individual terms $W_1^i \smile W_2^{r-i}$ have topological significance.

Reducing everything mod 2, write, for any \mathfrak{B} , the formal power series

$$W = \sum_i W^i t^i, \quad \bar{W} = 1/W = \sum_i \bar{W}^i t^i;$$

then

$$\bar{W}^0 = I|_2, \quad \bar{W}^1 = W^1, \quad \bar{W}^2 = W^2 + W^1 \smile W^1,$$

etc. The duality theorem gives then, as $W_N = W/W_T$, etc.,

$$W_N^r = \sum W^i \smile \bar{W}_T^{r-i}, \quad \bar{W}^r = \sum \bar{W}_T^i \smile \bar{W}_N^{r-i}, \text{ etc. (mod 2).}$$

7. *Tangent Bundles.*—Let K be a simplicial subdivision of the manifold M^n , with ordered vertices. Each p in K may be written uniquely as $p = \sum \eta_{\lambda_i}(p)x_{\lambda_i}$, if $p \in x_{\lambda_0} \dots x_{\lambda_r}$. Define

$$v_k(p) = \sum_{\lambda_0 < \dots < \lambda_k} \eta_{\lambda_0}(p) \dots \eta_{\lambda_k}(p)(x_{\lambda_k} - x_{\lambda_{k-1}}) \quad (k = 1, 2, \dots).$$

These are continuous in K , and the first r are independent except in K^{r-1} (any r). If K^* is the usual complex dual to K , these may be used to define W^r , a cocycle in K^* . Its dual is a characteristic cycle C^{n-r} in K' , the complex associated with \mathfrak{B} (which was studied in I, §13). Note that $C^{n-(2r+1)} = \frac{1}{2} \partial\omega C^{n-2r}$. The value of $C^s \cdot \sigma^s$ ($s = n - r, \sigma^s = x_{\lambda_0} \dots x_{\lambda_s}$) is as fol-

lows. Let K_1 be the subcomplex of the closed star of σ^s containing all vertices x_i with

$$\lambda_s > i > \lambda_{s-1} \text{ or } \lambda_{s-2} > i > \lambda_{s-3} \text{ or } \dots$$

(This includes all vertices below λ_0 if s is even.) Then $C^s \cdot \sigma^s = 1 - \chi(K_1)$ ($\chi =$ Euler-Poincaré characteristic), or this mod 2.

From this we prove: If K is the first derived of a simplicial subdivision of M , then C^s is the sum of all s -simplexes of K (properly oriented if integer coefficients are used).²

In the proofs of the following theorems, we study the classes over sub-manifolds of the given manifold, and use the duality theorem and results from §8. For $M^m \subset M^n$, let W^r mean the part of $W_T^r(M^n)$ in M^m .

If a closed M^m can be imbedded in E^n (with or without singularities), then

$$0 = W_N^{n-m}|_2 = \sum W^i \smile \overline{W}_T^{n-m-i} = \overline{W}_T^{n-m}.$$

Hence $\overline{W}_T^m = 0$ always. This gives, if $(X)^2 = X \smile X$, etc.,

$$\text{closed } M^2: W^2|_2 = (W^1)^2; \text{ closed } M^3: (W^1)^3 = 0;$$

$$\text{closed } M^4: W^4|_2 + (W^2)^2 + W^2 \smile (W^1)^2 + (W^1)^4 = 0; \text{ etc.}$$

In any M^2 , for any 1- I_2 -cocycle X^1 , $X^1 \smile X^1 \smile X^1 \smile W^1$. In any M^3 , $W^2 = W^1 \smile W^1$; hence (Stiefel) for orientable M^3 (closed or not), the tangent bundle is simple. In any orientable M^4 , $W^3 = 0$. (The proof uses facts from §10.) For any 2- I_2 -cocycle X^2 in any M^4 , $X^2 \smile X^2 \smile X^2 \smile \overline{W}_T^2$. For any orientable M^4 in an orientable M^r , $\overline{W}_T^4 = W_N^4$.

8. *Normal Bundles.*—For any M^m in any M^n , W_N^{n-m} is the intersection of M^m with itself in M^n , which is a cohomology class of the complex associated with the normal bundle; if M^m is closed and $M^n = E^n$, then $W_N^{n-m}|_2 = 0$, and if also M^m is orientable, then $W_N^{n-m} = 0$. Compare PC, §20, and TP, p. 795.

If M^m is mapped regularly into M^n , but with singularities, we may deform M^m slightly into M'^m , and consider the intersections of the σ' with a neighborhood of σ' in M ; then W^{n-m} is the *local intersection* thus defined. For a closed orientable $M^m \subset E^n$, the *intersection* vanishes, so that the *distant intersection* equals the local. This holds mod 2 in the non-orientable case. If $n = 2m$, and the singularities are isolated points, the distant intersection is of course 0 (mod 2); if m is odd, it vanishes, because $\{\sigma^m, \sigma'^m\} = -\{\sigma'^m, \sigma^m\}$.

Take an orientable $M^m \subset E^n$. Then the normal bundle is simple if $m = 1$ or 2, or $n = m + 1$ or $m + 2$, or $m = 3$ and M is closed, or M is a cell. This holds if M^m is merely mapped regularly, provided that if $n = m + 2$, then m is odd, and we omit $m = 2$ if M is closed.

9. *Examples.*—Consider a cylinder, the product $T^3 = T^1 \times T^2$ of a segment and a disc. Let P^2 be one end, let S_2^1 be a segment crossing P^2 ,

let Q^2 be a rectangle cutting through T^3 and ending on S_2^1 , and let S_1^1 be the center of Q^2 , ending at p_0 , the center of S_2 ; rather, let these be the sets after the identifications below. (1) Join the ends of T^3 , and shrink ∂T^2 to a point; this forms $M_1^3 = S^1 \times S^2$, with a simple tangent bundle. (2) Join the ends, and identify opposite points of ∂T^2 ; then characteristic cycles are (mod 2) $C^2 = Q^2$, $C^1 = S_1^1$; $\therefore \mathbf{W}^2 \neq 0$. (3) Join the ends, reflecting one so that P^2 is joined to itself with orientation reversed, and shrink ∂T^2 to a point. We obtain M_3^3 , with $\mathbf{W}^2 = 0$. (4) Join the ends as in (3), and identify opposite points of ∂T^2 , forming M_4^3 . Now P^2 is a projective plane, and Q^2 is a Klein bottle. Intersections are (mod 2)

$$\{P^2, P^2\} \sim 0, \{Q^2, P^2\} \sim S_2^1, \{Q^2, Q^2\} \sim S_1^1 + S_2^1;$$

$$\{P^2, S_2^1\} \sim 0, \{P^2, S_1^1\} \sim p_0, \{Q^2, S_2^1\} \sim p_0, \{Q^2, S_1^1\} \sim 0;$$

characteristic classes are

$$C^2 \sim (P^2 + Q^2)_2, C^1 \sim (S_1^1 + S_2^1)_2.$$

Define $\mathfrak{B}_1^1(M_4^3)$, with $C_1^2 \sim P^2$, $C_1^1 \sim 0$, and $\mathfrak{B}_2^1(M_4^3)$, with $C_2^2 \sim Q^2$, $C_2^1 \sim 0$; let the total spaces be M_1^4, M_2^4 . We may pretend M_4^3 is in either (because $C_i^1 = 0$). Then

$$C_1^3 \sim \mathfrak{S}(Q^2), C_1^2 \sim \mathfrak{S}(S_1^1), C_1^1 \sim \mathfrak{S}(p_0); \therefore \overline{C}_1^1 \sim 0;$$

$$C_2^3 \sim \mathfrak{S}(P^2), C_2^2 \sim \mathfrak{S}(S_2^1), C_2^1 \sim \mathfrak{S}(p_0); \therefore \overline{C}_2^1 \sim \mathfrak{S}(p_0);$$

hence in M_2^4 , $\mathbf{W}^3 \neq 0, \overline{\mathbf{W}}^3 \neq 0$. Hence (see §7) M_2^4 cannot be imbedded in E^7 .

Define $\mathfrak{B}^2(M_3^3)$, with $C^2 \sim P^2$, and $C^1 \sim S_1^1$. Then we may consider $M_3^3 \subset M^5 = \mathfrak{S}(M_3^3)$, and prove (a) M^5 is closed and orientable, (b) $C^2(M^5) \sim \mathfrak{S}(p)$; hence $\overline{\mathbf{W}}^3 = \mathbf{W}^3 \neq 0$, and M^5 cannot be imbedded in E^8 .

We may define $M^8 = \mathfrak{S}(S_0^4)$, with $\overline{\mathbf{W}}^4 = \mathbf{W}^4 \neq 0$; hence M^8 cannot be imbedded in E^{12} .

The complex projective plane P^{*4} cannot be imbedded in E^6 , as $\mathbf{W}^2 \neq 0$. ($\mathbf{W}^2 \smile \mathbf{W}^2 = \mathbf{W}^4|_2$; $W^4 \cdot P^{*4} = \chi(P^{*4}) = 3$.) But it can be in E^7 .

For any closed orientable M^4 , and any cocycles X^2, X^4 (mod 2) in M^4 , we may imbed M^4 in an $M^8 \subset E^{17}$, so that the part over M^4 of the characteristic classes of the normal bundle of $M^8 \subset E^{17}$ are X^2 and X^4 . Hence we may make $\mathbf{W}_N^3 \neq 0$ and $\mathbf{W}_N^4 \neq 0$ also.

If we put a Klein bottle Q^2 in E^3 , then the distant and local intersections are equal (mod 2); these are $\mathbf{W}_N^1 = \mathbf{W}_T^1$; C_T^1 is a closed curve in Q^2 . Hence the distant intersection, as a cycle, is a certain curve in Q^2 , as is clear in the usual immersion of Q^2 in E^3 . For P^2 in E^3 , we get the "projective line" similarly.

A direct study shows: If $M^2 \subset E^4$ is closed (with or without singularities), and C^2 is the fundamental cycle of the associated complex (integer

coefficients), then $W^2 \cdot C^2 = 2[\chi(M^2) + 2k]$ for some k ; any k may be obtained. Hence if $\chi \not\equiv 0 \pmod{2}$, then a field of normal vectors never exists. Also one imbedding cannot be deformed into another with a different $W^2 \cdot C^2$.

10. *Homology Groups of Total Spaces, Etc.*—Let K be connected. Given $\mathfrak{B}(K)$, the homology groups satisfy $\mathbf{H}^r(\mathfrak{S}) \approx \mathbf{H}^r(K)$, $r < \nu$. $kS(p) \sim 0$ if and only if for some $A^{\nu+1}$, $A^{\nu+1} \cdot W^{\nu+1} = k$. (We may use $k_\lambda = k \pmod{\lambda}$, and $A^{\nu+1}|_\lambda$.) For \mathfrak{B} oriented, a $(\nu+1)$ - I_μ -cycle A in K is the projection of such a cycle in \mathfrak{S} if and only if $A \cdot W^{\nu+1} = 0_\lambda$. Now $\mathbf{H}^r(\mathfrak{S})$ may be described in terms of $W^{\nu+1}$ and properties of $kS(p)$. If μ is the smallest integer such that $\mu S(p) \sim 0$, then $\mathbf{H}^r(\mathfrak{S}) \approx I_\mu \oplus \mathbf{H}^r(K)$ if and only if for each λ and each $(\nu+1)$ - I_λ -cycle A , $A \cdot W^{\nu+1} \equiv 0 \pmod{(\lambda, \mu)}$. A mapping f of a complex K' of dimension $\leq \nu+1$ into K is the projection of a mapping into \mathfrak{S} if and only if $f^*W^{\nu+1} = 0$.

¹ We refer the reader to papers in these PROCEEDINGS, 21, 464–468 (1935), and in *Bull. Am. Math. Soc.*, 43, 785–805. We denote the latter by TP, and the preceding note, by I. Sphere-bundles were formerly called “sphere-spaces.”

² This was a conjecture of Stiefel, *Comm. Math. Helv.*, 8, 40 (1936).