SOME ADVANCES IN THE COMBINATORY THEORY OF QUANTIFICATION

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The purpose of this paper is to report the present status of certain researches which have been interrupted by the war. These depend on a series of previous papers which will be referred to by abbreviations listed in a footnote.[1] The researches are those sketched in CFM, §§ 6–8. The notations of CFM and CCT will be adhered to, except as later specified.

1. *Preliminary conventions; the system* $\mathfrak{F}_0$. The basis of the investigation is that form of λ-formalism which has been proved equivalent to the system $\mathfrak{K}_2$ (CCT, pp. 57 ff.). To this are adjoined a unary predicate, expressed as usual by prefixed '$\vdash$', and an unspecified class of *canonical terms*. The rudimentary system so formed, with the rules stated below, will be called $\mathfrak{F}_0$.

Unspecified terms of $\mathfrak{F}_0$ and its extensions will be denoted by capital German letters; unspecified canonical terms by small Greek letters. When the statement of a postulate, theorem, rule, etc., involves such letters it is understood that arbitrary substitutions of appropriate terms can be made. The symbols '$\rightarrow$', '&', '$\equiv$' will be used as in PKR, § 2.4. The symbols '$=$', '$\geqq$' will be used, respectively, for 'is convertible to' and 'is reducible to', both in the sense of CCT, § 2.

The rules for $\mathfrak{F}_0$ (and its extensions) are: (1) if $\alpha = \mathfrak{B}$, then $\mathfrak{B}$ is canonical, and (2) the rule

RULE CONV.: $\vdash \mathfrak{A}$ & $\mathfrak{A} = \mathfrak{B} \rightarrow \vdash \mathfrak{B}$.

2. *The system* F_1. Suppose we adjoin to $\mathfrak{F}_0$: (1) the primitive term F, (2) the rule given for F in CFM, which will be called RULE F, (3) the rule that $F\alpha\beta$ is always canonical, and (4) Postulates (*FK*) and (*FS*) of CFM. The system so formed will be called $\mathfrak{F}_1$.

In $\mathfrak{F}_1$ all the theorems of FPF, §§ 2, 3, 4 are valid if appropriate changes (mostly replacing quantified variables by Greek letters and taking implications metatheoretically) are made. However, the proof of such theorems as 3.11 and 3.12 is facilitated by the following.

THEOREM. *If* $\mathfrak{X}$ *is a combination of the variables* $x_1, \ldots, x_n$ *and the terms* $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$, *and if* $\alpha_1, \ldots, \alpha_m$ *are such that*

$$\vdash \alpha_i \mathfrak{A}_i \qquad i = 1, 2, \ldots, m; \tag{1}$$

if, further, it follows from the assumptions

$$\vdash \xi_j x_j \qquad j = 1, 2, \ldots, n$$

and the theorems (1), by following out the construction of $\mathfrak{X}$ *and applying Rule F, that*

$$\vdash \eta\mathfrak{X};$$

then without hypothesis

$$\vdash F_n\xi_1 \ldots \xi_n\eta(\lambda^n x_1 \ldots x\mathfrak{X}_n).$$

This theorem can be proved directly by induction with greater ease than most of the theorems of FPF, § 3.

The consistency of $\mathfrak{F}_1$ follows readily provided only that canonical terms are such that $\alpha\mathfrak{X}$ can reduce only to terms of the form $\alpha'\mathfrak{X}'$ where $\alpha \geqq \alpha'$ and $\mathfrak{X} \geqq \mathfrak{X}'$.[2]

3. The system $\mathfrak{F}_2$. To form the system $\mathfrak{F}_2$ adjoin Ξ to $\mathfrak{F}_0$ as new primitive term and also the rule, henceforth called RULE Ξ, given for Ξ in CFM. We also need the rule that $\Xi\alpha\beta$, $\alpha\mathfrak{X}$ and $\lambda x\alpha$, are canonical.[3] Next define

$$\mathfrak{A} \supset_x \mathfrak{B} \equiv \Xi(\lambda x\mathfrak{A})(\lambda x\mathfrak{B}).$$

The postulates are then:

POST. (ΞK). $\vdash \alpha x \supset_x \cdot \beta xy \supset_y \alpha x$

POST. (ΞS). $\vdash \alpha u \supset_u \cdot \beta uv \supset_v \gamma xuv \cdot \supset_x : \ \alpha w \supset_w \beta_w(yw) \cdot \supset_y \cdot \alpha z \supset_z \gamma xz(yz).$

If F is defined by (10) of CFM, this system includes $\mathfrak{F}_1$. The following deduction theorem can also be proved:

THEOREM. *If from the assumption that*

$$\vdash \xi x,$$

together with certain axioms which do not contain x, it is derivable by the rules of $\mathfrak{F}_2$ *that*

$$\vdash \mathfrak{X};$$

then from these same axioms it is derivable absolutely within $\mathfrak{F}_2$ *that*

$$\vdash \xi x \supset_x \mathfrak{X},$$

provided that the axioms and the terms corresponding to $\mathfrak{A}$, $\mathfrak{B}$ *in every application of Rule* Ξ *are canonical.*

The consistency theorem of § 5 removes the last proviso. On the other hand, the theorem, apparently, cannot be iterated to take care of multiple premises such as $\vdash \xi x$, $\vdash \eta xy$; for when we eliminate one premise we introduce new axioms, derived from the postulates of $\mathfrak{F}_2$, which may contain other variables. This shows that the postulates for $\mathfrak{F}_2$ are still in a tentative state.

4. *The system* $\mathfrak{F}_3$. The system obtained by adjoining to $\mathfrak{F}_0$ the terms Π and P with the rules set down for them in CFM will be called $\mathfrak{F}_3$. For canonical terms we must have $\Pi\alpha$, $P\alpha\beta$, $\alpha\mathfrak{X}$ and $\lambda x\alpha$.[4] The postulates will be some as yet undetermined set sufficiently strong to include $\mathfrak{F}_2$ (with Ξ defined as in (8) of CFM) and the postulates stated in CFM.

5. *Combinatory verifiability; the systems* $\mathfrak{B}$. For the study of the consistency of $\mathfrak{F}_2$ and $\mathfrak{F}_3$ it is expedient to introduce systems $\mathfrak{B}_2$ and $\mathfrak{B}_3$ which have the same relation to the former that the Gentzen system LJ has to the Heyting calculus. These involve two predicates of an undetermined number of arguments, viz.,

$$\text{Can}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}) \qquad n \geqq 0 \tag{1}$$

$$\text{Ver}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}) \qquad n \geqq 0. \tag{2}$$

The interpretations of the formulas (1), (2) are "$\mathfrak{A}$ is canonical (or verifiable, respectively) on the basis $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$"; when $n = 0$ this will mean simply that the term $\mathfrak{A}$ is canonical or verifiable. The formula

$$\text{Reg}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}) \qquad n \geqq 0. \tag{3}$$

will be used to stand for either (1) or (2). A sequence of terms to which (1), (2) or (3) applies will be called a canonical, verifiable or regular sequence, respectively; in such a sequence $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ are the antecedents and $\mathfrak{A}$ the consequent.

In the following statement of the rules '$\binom{\mathfrak{U}}{x}\mathfrak{A}$' means the result of substituting $\mathfrak{U}$ for x in $\mathfrak{A}$; the accents should be ignored for the present.

I. $\text{Can}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}') \rightarrow \text{Ver}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}, \mathfrak{A})$.

II. *If* $k_1, \ldots, k_r$ *is any sequence such that each* k_j *has one of the values* $1, 2, \ldots, n$, *then*

$$\text{Can}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_{n-1}, \mathfrak{M}_n') \ \&\ \text{Reg}\ (\mathfrak{M}_{k_1}, \ldots, \mathfrak{M}_{k_r}, \mathfrak{A}) \rightarrow \text{Reg}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}).$$

III. *If* $\mathfrak{A} \geqq \mathfrak{B}$, $\mathfrak{M}_k \geqq \mathfrak{N}_k$ *then*

$$\text{Reg}\ (\mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{B}) \rightarrow \text{Reg}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}).$$

IV 1. *If* x *does not occur in* $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$, *then*

$$\text{Reg}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}, \mathfrak{B}) \rightarrow \text{Reg}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A} \supset_x \mathfrak{B}).$$

IV 2. *If* x *does not occur in* $\mathfrak{M}_1, \ldots, \mathfrak{M}_m$, *then*

$$\text{Can}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \mathfrak{A}, \mathfrak{B}') \ \&\ \text{Ver}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \binom{\mathfrak{U}}{x}\mathfrak{A}) \ \&\ \text{Reg}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \binom{\mathfrak{U}}{x}\mathfrak{B}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{C}) \rightarrow \text{Reg}\ (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A} \supset_x \mathfrak{B}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{C}).$$

These are the rules for $\mathfrak{B}_2$; for $\mathfrak{B}_3$ IV should be replaced by analogous rules for Π and P.

On this basis it can be shown that: (1) if a sequence is canonical, so is the subsequence obtained by omitting the consequent; (2) every verifiable sequence is canonical; (3) arbitrary substitutions can be made for the variables; (4) a theorem roughly the converse of IV holds; and (5) there is a theorem, called the *elimination theorem*, which is analogous to Gentzen's "Schnitt"—in particular it allows an antecedent to be dropped when it is verifiable on the basis of its predecessors. Also (6) canonicalness is invariant with respect to conversion.

These theorems presuppose that there are certain elementary canonical terms beginning with constants other than S, K, for which terms there are specifications consistent with the theorems. An example of such a specification is: Can $(Q\mathfrak{X}\mathfrak{Y})$, Ver $(Q\mathfrak{X}\mathfrak{X})$. Every term has a certain finite order, with reference to constructions by means of Ξ from these elementary constituents; the proof of the elimination theorem involves an induction with respect to this order.

In applying this argument to the consistency of $\mathfrak{F}_2$ and $\mathfrak{F}_3$ certain changes in the definitions of canonicalness for these systems have to be made.[5] When this is done we have:

THEOREM. *If the axioms of the systems* $\mathfrak{F}_2$ *and* $\mathfrak{F}_3$ *are verifiable, then all the theorems are verifiable.*

Since S, K, Ξ, Π, etc., are not canonical and QSK (on the above assumption) is canonical but not verifiable, the consistency of $\mathfrak{F}_2$ and $\mathfrak{F}_3$ is proved.

6. *The systems* $\mathfrak{F}^*$. We consider now the problem of expressing the "postulates" of the systems $\mathfrak{F}$, which are axiom schemes, as single axioms. There are two ways of doing this: (1) by introducing quantifiers of higher order; (2) by postulating a term H, representing canonicalness, and generalizing with the quantifiers already present. The former course is not investigated here at all; the second only partially. Certain remarks concerning it are as follows.

First there must be a notion of relative canonicalness. It is not sufficient to have $\Xi\mathfrak{A}\mathfrak{B}$ canonical when $\mathfrak{A}$ and $\mathfrak{B}$ are; there must be circumstances under which $\mathfrak{A}$ is verifiable only for certain values of x, and $\mathfrak{B}$ canonical for the same values.

Secondly, one cannot postulate $\vdash H(H\mathfrak{X})$. For on the assumption as to relative canonicalness made below, there would then be, for any canonical $\mathfrak{B}$, a canonical term $\mathfrak{A}$ such that $\mathfrak{A} = H\mathfrak{A} \supset \cdot \mathfrak{A} \supset \mathfrak{B}$, and from this a contradiction arises much as in IFL.[6] The system is inconsistent if $\vdash H^n\mathfrak{X}$ is postulated for any specific n.[7]

Thirdly, there seems to be no use in considering $\mathfrak{F}_3^*$. For if Ξ is expressible only through Π and P, then from $\vdash \Xi H\mathfrak{A}$ we must infer $\vdash P(H\mathfrak{X})(\mathfrak{A}\mathfrak{X})$ for any $\mathfrak{X}$, even when neither $H\mathfrak{X}$ nor $\mathfrak{A}\mathfrak{X}$ is canonical. It is difficult to see how we can get a verifiability theorem in such a case. I suspect that $\mathfrak{F}_3$* (on any reasonable formulation) is inconsistent.

On the other hand, it is probably possible to formulate the system $\mathfrak{F}_2$* and to prove its consistency by constructing an appropriate $\mathfrak{B}$* (see § 7). Moreover, this can presumably be done so as to remove the difficulties in regard to $\mathfrak{F}_2$ mentioned in § 3. For conjectures as to the possible significance of this for mathematics see CFM, § 8.

7. *The system* $\mathfrak{B}_2$*. The formulation of the system $\mathfrak{B}_2$* will not be given here in detail. However, the changes which must be made in the formulation of § 5 are essentially as follows:

1. Care must be used with variables, since now substitutions for variables cannot be made except when certain premises are fulfilled. We must distinguish three kinds of variables. A variable 'x' occurring in a specific place in a term of a sequence shall be said to be *apparent* there if that place is part of the scope of a prefix 'λx', otherwise it is *real;* if it is real but is also real in some previous term of the sequence it shall be said to be *bound;* otherwise it is *free.* Substitutions are not permitted for any of these—such substitutions are taken care of by the elimination theorem. It may be desirable to have a fourth class of variables, called *indeterminates,* for which substitutions can be made. It is advisable to use different classes of letters for indeterminates and apparent variables than those which are used for real variables. Some changes to this effect should be made in IV.

2. Where the consequents of certain sequences in § 5 are primed it is to be understood that there are values for the free variables for which the formula holds. These values may be functions of the bound variables and indeterminates.

3. The suggested rules for H are

V 1. Can $(\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}) \rightarrow$ Reg $(\mathfrak{M}_1, \ldots, \mathfrak{M}_n, H\mathfrak{A})$.

V 2. *If* (*a*) Can $(\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \mathfrak{A}')$ (1)

and if (*b*) *from* Can $(\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \mathfrak{A})$ (2)

it is deducible[8] *that*

$$\text{Reg } (\mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{B}) \quad (3)$$

then

$$\text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_m, H\mathfrak{A}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{B}). \quad (4)$$

4. The form of the rule V requires reformulation of the concept of a proof. If we call an ordinary proof a deduction—which is a chain of formulas connected according to the rules—we have here to do with a chain of deductions, where a whole deduction of lower order can be premise

for a formula in one of higher order. The exact formulation of this requires considerable space.

5. The induction used in the proof of the elimination theorem will be of a more complicated nature.

Can the elimination theorem be proved from these premises, or some suitable modifications of them, by a finite or transfinite induction? On that question research is unfinished.

[1] These abbreviations consist of the initials of the first three principal words of the title. The citations are

CCT. *Jour. Symbolic Logic,* **6,** 54–61 (1941).

CFM. *Jour. Symbolic Logic,* **7,** 49–64 (1942).

FPF. *Tôhoku Math. Jour.,* **41,** 371–401 (1936).

IFL. "*Jour. Symbolic Logic,* **7,** 115–117 (1942).

PKR. *Trans. Amer. Math.* **50,** 454–516 (1941).

[2] For it can be shown that if $\vdash \mathfrak{X}$, then $\mathfrak{x}$ reduces to a term having that relation to a formula of the algebra of pure implication which was noted in CFM.

[3] This is an over-simplification. In $\mathfrak{F}_{\lambda 2}$ it would be more appropriate to have different ranks of canonicalness with the rules: (1) If α and β are canonical of rank 1, then $\Xi\alpha\beta$ is canonical of rank 0; (2) if α is canonical of rank n then $\lambda x\alpha$ is canonical of rank $n + 1$ and, if $n > 0$, $\alpha\mathfrak{X}$ is canonical of rank $n - 1$. Cf. end of § 5.

[4] Cf. the preceding footnote.

[5] Cf. footnote [3].

[6] One can, of course, introduce notions of canonicalness of higher order. For conjectures relative to this see CFM, § 8.

[7] Note H^n here has the significance given to the nth power by Rosser, *Annals of Mathematics,* **36,** 129 (1935).

[8] The free variables of (2) being carried as indeterminates.